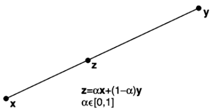
Chapter 4 Concepts from Geometry

1. Line Segments
2. Hyperplanes and Linear Varieties
3. Convex Sets
4. Neighborhoods
5. Polytopes and Polyhedra

Line Segments

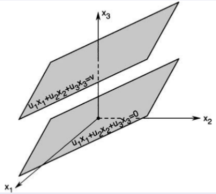
Definition (line segment)

Line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ can be represented as $\{\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \mid \alpha \in [0,1]\}$.



Definition (hyperplane: for a $0 \neq \boldsymbol{u} \in \mathbb{R}^n$ and $v \in \mathbb{R}$)

Hyperplane is defined by
$\boldsymbol{\pi} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^\top \boldsymbol{x} = v\}$ or
$\boldsymbol{\pi} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^\top (\boldsymbol{x} - \boldsymbol{a}) = 0\}$, $\boldsymbol{a} \in \boldsymbol{\pi}$.
$\boldsymbol{u}$ is called normal vector of hyperplane.
If $v = 0$, then $0 \in \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}$ is a subspace.
If $v \neq 0$, then $0 \notin \boldsymbol{\pi} \Longrightarrow \boldsymbol{\pi}$ is a manifold.



- positive halfspaces: $\boldsymbol{H}^+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^\top (\boldsymbol{x} - \boldsymbol{a}) \geq 0\}$,
- negative halfspaces: $\boldsymbol{H}^- = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^\top (\boldsymbol{x} - \boldsymbol{a}) \leq 0\}$.

Linear Variety

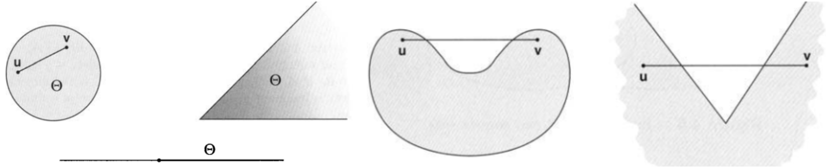
Definition (linear variety: $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

Linear variety is $\boldsymbol{L} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$. The dimension of $\boldsymbol{L}$ is defined as $\dim\mathcal{N}(\boldsymbol{A})$. $\boldsymbol{L}$ is a subspace $\iff$ $\boldsymbol{b} = 0$.

- If $\boldsymbol{A} = 0$, then $\boldsymbol{L} = \mathbb{R}^n$.
- If $\dim\boldsymbol{L} < n$, then $\boldsymbol{L}$ is the intersection of a finite number of hyperplanes.

Definition (convex set: for a set $\Theta \subset \mathbb{R}^n$)

- Θ is convex $\iff$ $\forall \boldsymbol{x}, \boldsymbol{y} \in \Theta$, line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ is in Θ.
- Θ is convex $\iff$ $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in \Theta$, $\forall \boldsymbol{x}, \boldsymbol{y} \in \Theta$ and $\alpha \in (0,1)$.



Convex Set

★ Some examples of convex set: empty set, single point, line, line segment, subspace, hyperplane, linear variety, half-space, polyhedron, second order cone, positive semidefinite cone, $\cdots$

how to check a set is convex or not? (proof on blackboard)

- $\Theta = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$.
- $\Theta = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x} + c \leq 0, \boldsymbol{A} \succ 0\}$.

Theorem (preserving convexity:)

If Θ, Θ_1 and Θ_2 are convex sets in $\mathbb{R}^n$, then

- *scaling:* $r\Theta = \{r\boldsymbol{x} \mid \forall \boldsymbol{x} \in \Theta\}$;
- *Minkowski sum/difference:* $\Theta_1 \pm \Theta_2 = \{\boldsymbol{x}_1 \pm \boldsymbol{x}_2 \mid \forall \boldsymbol{x}_1 \in \Theta_1, \boldsymbol{x}_2 \in \Theta_2\}$;
- *Linear transform:* $\boldsymbol{A}\Theta = \{\boldsymbol{A}\boldsymbol{x} \mid \forall \boldsymbol{x} \in \Theta\}$.

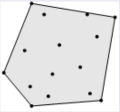
Convex/Conic/Affine Hull

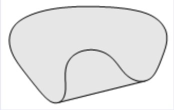
convex hull of C: conv(C)

- discrete set $C = \{\boldsymbol{x}_i\}_{i=1}^k$:

conv(C)=$\{\sum_{i=1}^k \alpha_i \boldsymbol{x}_i \mid \sum_{i=1}^k \alpha_i = 1,\ \alpha_i \geq 0\}$.

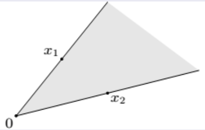
- continuous set C: conv(C) is all convex combination of points in C.





conic hull of C: cone(C)

- discrete set $C = \{\boldsymbol{x}_i\}_{i=1}^k$: cone$(C)$=$\left\{\sum_{i=1}^k \alpha_i \boldsymbol{x}_i \mid \alpha_i \geq 0\right\}$.
- continuous set C: cone(C) is all conic combination of points in C.



affine hull of C: aff(C)

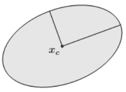
affine hull of C is the intersection of all affine sets containing C. aff(C) is a set of the form $\boldsymbol{x} + V$ with V as a subspace.

Euclidean ball with center $\boldsymbol{x}_c$ and radius r

$$\mathcal{B}(\boldsymbol{x}_c, r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \leq r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \leq 1\}$$

ellipsoid with center $\boldsymbol{x}_c$ and radius r

- $\mathcal{E} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_c\|_{\boldsymbol{P}}^2 \leq r\}$ with $\boldsymbol{P} \succ 0$ and $\boldsymbol{P} = \boldsymbol{P}^\top$
- $\mathcal{E} = \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \leq r\}$ with $\boldsymbol{A}$ is square and nonsingular

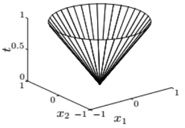


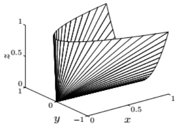
cone

- norm cone: $K = \{(\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|\boldsymbol{x}\|_2 \leq t\}$
- positive semidefinite cone:

$\mathbb{S}_+^n = \{\boldsymbol{A} \in \mathbb{R}^{n \times n} \mid \boldsymbol{A} = \boldsymbol{A}^\top, \ \boldsymbol{A} \succeq 0\}$

For example, $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}_+^2$





Definition (extreme point of a set $\Theta \subset \mathbb{R}^n$)

A point $\boldsymbol{x} \in \Theta$ is said to be an extreme point if there are no two distinct points $\boldsymbol{u}$ and $\boldsymbol{v}$ in Θ such that $\boldsymbol{x} = \alpha\boldsymbol{u} + (1-\alpha)\boldsymbol{v}$ for some $\alpha \in (0,1)$.

★ draw line segments at $\boldsymbol{x}$, if all line segments doesn't belong to Θ, then $\boldsymbol{x}$ is extreme point.

Example (extreme point)

- Disk: any point on the boundary; Polyhedron: vertex;
- half-line: endpoint of the half-line.

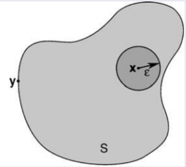
Definition (neighborhoods)

For an $\varepsilon > 0$, a neighborhood of $\boldsymbol{x} \in \mathbb{R}^n$ is $N(\boldsymbol{x}) = \{\boldsymbol{x}' \in \mathbb{R}^n \mid \|\boldsymbol{x}' - \boldsymbol{x}\| \leq \varepsilon\}$, i.e., $N(\boldsymbol{x})$ is a norm ball with radius ε and center $\boldsymbol{x}$.

Neighborhoods

Definition (interior/boundary/relinterior of $S \subset \mathbb{R}^n$. An $\boldsymbol{x} \in \mathbb{R}^n$ is)

• interior point of $S \iff \exists\ N(\boldsymbol{x}) \subseteq S$.
• boundary point of $S \iff \forall\ N(\boldsymbol{x})$ contains point in S and $\mathbb{R}^n \backslash S$.
• relative interior point of $S \iff \boldsymbol{x}$ is an interior point of S relative to aff(S), i.e., $\exists\ N(\boldsymbol{x}) \cap \text{aff}(S) \subseteq S$.



★ The interior of S, denoted by int$S \iff$ all interior points of S.

★ A boundary point may not be an element of S. All boundary points of S is called the boundary of S, denoted by bd(S).

Definition (A set $S \subset \mathbb{R}^n$ is open/closed/bounded/compact)

- open $\iff$ all its points are interior point.
- closed $\iff$ it contains its boundary $\iff$ its complement is open.
- bounded $\iff$ it is contained in a ball of finite radius, $\|\boldsymbol{x}\| \leq r$, $\forall \boldsymbol{x} \in S$.
- compact $\iff$ it is both closed and bounded.

Example

A concave function $f : \mathbb{R}^n \to \mathbb{R}$, which attains its minimum over a convex set X at an $\boldsymbol{x}^* \in \text{ri}(X)$, must be constant over X.

proof. $\because$ $\boldsymbol{x}^* \in \text{ri}(X)$.

$\therefore$ there exist $\boldsymbol{x} \in X$ such that prolong beyond $\boldsymbol{x}^*$ the line segment $\boldsymbol{x}$-to-$\boldsymbol{x}^*$ to a point $\bar{\boldsymbol{x}} \in X$.

i.e., $\exists\ \alpha \in (0, 1)$ such that $\boldsymbol{x}^* = \alpha\boldsymbol{x} + (1-\alpha)\bar{\boldsymbol{x}}$.

By concavity of f, we have

$$\begin{aligned} f(\boldsymbol{x}^*) = f(\alpha\boldsymbol{x} + (1-\alpha)\bar{\boldsymbol{x}}) &\geq \alpha f(\boldsymbol{x}) + (1-\alpha)f(\bar{\boldsymbol{x}}) \\ &\geq \alpha f(\boldsymbol{x}^*) + (1-\alpha)f(\boldsymbol{x}^*) = f(\boldsymbol{x}^*), \end{aligned}$$

we have "$=$" holds in above inequality.

$\therefore$ $f(\boldsymbol{x}^*) = (\boldsymbol{x}) = f(\bar{\boldsymbol{x}})$.

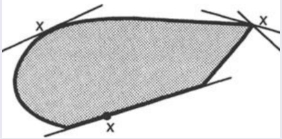
Theorem (Weierstrass theorem)

Let $\Omega \subset \mathbb{R}^n$ be a compact set and $f : \Omega \to \mathbb{R}$ be a continuous function. Then, there exists an $\boldsymbol{x}_0 \in \Omega$ such that $f(\boldsymbol{x}_0) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$, i.e., f achieves its minimum on Ω.

- ★ Weierstrass Theorem implies the existence of solution $\min\limits_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})$.
- ★ Strict convexity, lowersemi continuous, coercive, can guarantee the uniqueness of solution $\min\limits_{\boldsymbol{x}\in\Omega} f(\boldsymbol{x})$.

Definition (supporting hyperplane)

Let $\Theta \subset \mathbb{R}^n$ be convex and $\boldsymbol{x} \in \mathrm{bd}(\Theta)$. A hyperplane $\boldsymbol{\pi}$ passing through $\boldsymbol{x}$ is called a supporting hyperplane of Θ if the entire set Θ lies completely in one of the half-spaces divides by hyperplane $\boldsymbol{\pi}$.

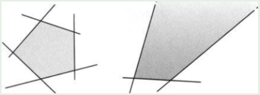


Definition (polytopes and polyhedra)

- convex polytope: the intersection of a finite number of halfspaces.
- polyhedron: nonempty bounded polytope.

Example

both are convex polytope. The first one is polyhedron.



Definition (carrier of polyhedron $\Theta \subset \mathbb{R}^n$)

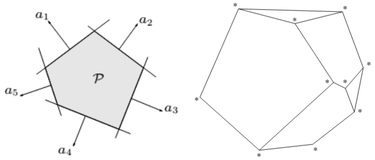
$\exists$ an integer $0 \le k \le n$ such that Θ is contained in a linear variety of dimension k, but is not entirely contained in any $(k-1)$-dimensional linear variety. Furthermore, $\exists \mid k$-d linear variety containing Θ, called the carrier of Θ, and k is called the dimension of Θ.

Polytopes and Polyhedra

Definition (vertex/edge/face of polyhedron)

$P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = b, \boldsymbol{C}\boldsymbol{x} \leq d\}$

- face of P: $(k-1)$-dimensional polyhedra forming the boundary of a k-dimensional polyhedron. Every k-dimensional polyhedron has faces of dimensions $k-1$, $k-2$, $\cdots$.
- vertex of P: 0-dimensional face of a polyhedron.
- edge of P: 1-dimensional face of a polyhedron.



Homework: Exercise in Textbook: 4.3