Chapter 3　Transformations

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Linear Transformations

Definition (linear transformations)

A function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if:

1. $\mathcal{L}(a\boldsymbol{x}) = a\mathcal{L}(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.
2. $\mathcal{L}(\boldsymbol{x} + \boldsymbol{y}) = \mathcal{L}(\boldsymbol{x}) + \mathcal{L}(\boldsymbol{y})$ for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

Definition (matrix representation)

Suppose that $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{x}'$ is the representation of $\boldsymbol{x}$ with respect to the given basis for $\mathbb{R}^n$. If $\boldsymbol{y} = \mathcal{L}(\boldsymbol{x})$, and $\boldsymbol{y}'$ is the representation of $\boldsymbol{y}$ with respect to the given basis for $\mathbb{R}^m$, then $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{x}'$. $\boldsymbol{A}$ is called the matrix representation of $\mathcal{L}$ with respect to the given bases for $\mathbb{R}^n$ and $\mathbb{R}^m$.

★ Particularly, for the natural bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, the matrix representation $\boldsymbol{A}$ satisfies $\mathcal{L}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$.

Linear Transformations

Definition (transformation matrix)

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ and $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n'\}$ be two bases for $\mathbb{R}^n$. Define the matrix $\boldsymbol{T} = [\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n']^{-1}[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n]$, or equivalently

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n] = [\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n']\boldsymbol{T},$$

$\boldsymbol{T}$ is called the transformation matrix.

Example

For any $\boldsymbol{u} \in \mathbb{R}^n$, let $\boldsymbol{x}$ (resp. $\boldsymbol{x}'$) be the coordinates of $\boldsymbol{u}$ with respect to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ (resp. $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n'\}$). Then, $\boldsymbol{x}' = \boldsymbol{T}\boldsymbol{x}$.

Similarity: A linear transformation $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}^m$.

Let $\boldsymbol{A}$ (resp. $\boldsymbol{B}$) be its representation of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ (resp. $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_n'\}$).

Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{y}' = \boldsymbol{B}\boldsymbol{x}'$.

$\therefore\ \boldsymbol{y}' = \boldsymbol{T}\boldsymbol{y} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}' = \boldsymbol{B}\boldsymbol{T}\boldsymbol{x}$,

hence, $\boldsymbol{T}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{T}$, or $\boldsymbol{A} = \boldsymbol{T}^{-1}\boldsymbol{B}\boldsymbol{T}$.

Eigenvalue and Eigenvector

Definition (eigenvalue and eigenvector)

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. A scalar $\lambda \in \mathbb{C}$ and a vector $\boldsymbol{v} \neq 0$ satisfying $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$ are said to be an eigenvalue and an eigenvector of $\boldsymbol{A}$.

★ Calculation of eigenvalues/spectrum of $\boldsymbol{A}$ $\Longleftrightarrow$
$\det[\lambda \boldsymbol{I} - \boldsymbol{A}] = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$ (characteristic equation)

Corollary

If $\det[\lambda \boldsymbol{I} - \boldsymbol{A}] = 0$ *has* n *distinct roots* $\{\lambda_i\}_{i=1}^n$. *Then, there exist* n *linearly independent vectors* $\{\boldsymbol{v}_i\}_{i=1}^n$ *such that* $\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, $i = 1, \ldots, n$.

Theorem (let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a square matrix)

- $\boldsymbol{A}$ *is similar to diagonal matrix* $\Longleftrightarrow$ $\boldsymbol{A}$ *has* n *linearly independent eigenvectors* $\{\boldsymbol{v}_i\}_{i=1}^n$.
- $\boldsymbol{A}$ *is similar to diagonal matrix* $\Longleftarrow$ $\boldsymbol{A}$ *has* n *distinct eigenvalues* $\{\lambda_i\}_{i=1}^n$.

Eigenvalue and Eigenvector

procedures for diagonalizing a matrix

1. calculate the eigenvalues of $\boldsymbol{A}$, i.e., $\{\lambda_i\}_{i=1}^n$;
2. calculate the eigenvectors of $\boldsymbol{A}$, i.e., $\{\boldsymbol{v}_i\}_{i=1}^n$;
3. let $\boldsymbol{T} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n]$, $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1}$.

Theorem (symmetric matrix: $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ satisfying $\boldsymbol{A} = \boldsymbol{A}^\top$)

- *All eigenvalues of a real symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ are real.*
- *Any real symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has a n mutually orthogonal eigenvectors. (proof on blackboard)*

Definition (orthogonal matrix)

A matrix whose transpose is its inverse is said to be an orthogonal matrix, i.e., $\boldsymbol{T}^{-1} = \boldsymbol{T}^\top$.

Eigenvalue and Eigenvector

Theorem (diagonalize symmetric matrix)

Any real symmetric matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ *has a diagonal form* $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^\top$ *with* $\boldsymbol{T}$ *as orthogonal matrix and* $\boldsymbol{\Lambda}$ *as diagonal matrix.*

Procedures for diagonalizing a real symmetry matrix

1. calculate the eigenvalues of $\boldsymbol{A}$, i.e., $\{\lambda_i\}_{i=1}^n$;
2. calculate the eigenvectors of $\boldsymbol{A}$, i.e., $\{\boldsymbol{v}_i\}_{i=1}^n$;
3. normalize individually the eigenvectors of λ_i, $\boldsymbol{T} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n]$
4. let $\boldsymbol{T} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n]$, $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, then $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^\top$.

Orthogonal Projections

Definition (subspace)

A set $\mathcal{V} \subseteq \mathbb{R}^n$ a subspace if $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in \mathcal{V} \Longrightarrow \alpha \boldsymbol{x}_1 + \beta \boldsymbol{x}_2 \in \mathcal{V}$, $\forall \alpha, \beta \in \mathbb{R}$. The dimension of $\mathcal{V}$, denoted by dim$\mathcal{V}$, is the maximum number of linearly independent vectors in $\mathcal{V}$.

Definition (orthogonal complement)

If $\mathcal{V} \subseteq \mathbb{R}^n$ a subspace, then the orthogonal complement of $\mathcal{V}$, denoted by $\mathcal{V}^\perp$, consists of all vectors that are orthogonal to every vector in $\mathcal{V}$, i.e., $\mathcal{V}^\perp = \{\boldsymbol{x} \mid \boldsymbol{v}^\top \boldsymbol{x} = 0, \forall \boldsymbol{v} \in \mathcal{V}\}$.

★ $V^\perp$ is also a subspace of $\mathbb{R}^n$. $\mathcal{V}$ and $\mathcal{V}^\perp$ span $\mathbb{R}^n$ (or $\mathbb{R}^n$ is the direct sum of $\mathcal{V}$ and $\mathcal{V}^\perp$), i.e., $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{V}^\perp$. Concisely, every $\boldsymbol{x} \in \mathbb{R}^n$ can be represented uniquely as

$$\text{[orthogonal decomposition] } \boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \text{where } \boldsymbol{x}_1 \in \mathcal{V}, \boldsymbol{x}_2 \in \mathcal{V}^\perp.$$

★ $\boldsymbol{x}_1$ (resp. $\boldsymbol{x}_2$) is orthogonal projections of $\boldsymbol{x}$ onto $\mathcal{V}$ (resp. $\mathcal{V}^\perp$).

Orthogonal Projections

Definition (orthogonal projections)

A linear transformation $\boldsymbol{P}$ is an orthogonal projector onto $\mathcal{V}$ if for all $\boldsymbol{x} \in \mathbb{R}^n$, we have $\boldsymbol{P}\boldsymbol{x} \in \mathcal{V}$ and $\boldsymbol{x} - \boldsymbol{P}\boldsymbol{x} \in \mathcal{V}^{\perp}$.

Definition (range and null space of matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$)

- The range (or image) of $\boldsymbol{A}$: $\mathcal{R}(\boldsymbol{A}) = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\}$;
- The nullspace (or kernel) of $\boldsymbol{A}$: $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = 0\}$.

★ Both $\mathcal{R}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{A})$ are subspaces.

Lemma

For matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\mathcal{R}(\boldsymbol{A})^{\perp} = \mathcal{N}(\boldsymbol{A}^{\top})$ and $\mathcal{N}(\boldsymbol{A})^{\perp} = \mathcal{R}(\boldsymbol{A}^{\top})$. (proof on blackboard)

Orthogonal Projections

Theorem (property of projection)

A matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ *is an orthogonal projector onto the subspace* $\mathcal{V} = \mathcal{R}(\boldsymbol{P})$ $\iff$ $\boldsymbol{P}^2 = \boldsymbol{P} = \boldsymbol{P}^\top$. *(proof on blackboard)*

Definition (quadratic form)

$f : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form $\iff$ $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}$ with $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$.

★ w.l.o.g, $\boldsymbol{Q}$ is assumed to be symmetric. If $\boldsymbol{Q}$ is asymmetric, how?

Quadratic Form

A quadratic form $\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}$ is said to be

$$\begin{aligned}
\text{positive definite} &\iff \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} > 0,\ \forall \boldsymbol{x} \neq 0 &\iff \boldsymbol{Q} \succ 0;\\
\text{positive semidefinite} &\iff \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} \geq 0,\ \forall \boldsymbol{x} \neq 0 &\iff \boldsymbol{Q} \succeq 0;\\
\text{negative definite} &\iff \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} < 0,\ \forall \boldsymbol{x} \neq 0 &\iff \boldsymbol{Q} \prec 0;\\
\text{negative semidefinite} &\iff \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} \leq 0,\ \forall \boldsymbol{x} \neq 0 &\iff \boldsymbol{Q} \preceq 0.
\end{aligned}$$

Let $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, and $\boldsymbol{Q}_p = \begin{bmatrix} q_{i_1 j_1} & q_{i_1 j_2} & \cdots & q_{i_1 j_p} \\ q_{i_2 j_1} & q_{i_2 j_2} & \cdots & q_{i_2 j_p} \\ \vdots & \vdots & \ddots & \vdots \\ q_{i_p j_1} & q_{i_p j_2} & \cdots & q_{i_p j_p} \end{bmatrix}, \quad \begin{matrix} 1 \leq i_1 \leq \cdots \leq i_p \leq n, \\ 1 \leq j_1 \leq \cdots \leq j_p \leq n. \end{matrix}$

Definition (minor, principal minor, leading principal minor)

- p-order minor of $\boldsymbol{Q}$: $\det \boldsymbol{Q}_p$.
- p-order principal minor of $\boldsymbol{Q}$: $\det \boldsymbol{Q}_p$ with $i_k = j_k$ for all $k = 1, \ldots, p$.
- p-order leading principal minors of $\boldsymbol{Q}$: $\det \boldsymbol{Q}_p$ with $i_k = j_k = k$ for $k = 1, \ldots, p$.

Quadratic Form

How to check a symmetric matrix $\boldsymbol{Q}$ is positive definite?

1. $\boldsymbol{Q} \succ 0 \iff$ all eigenvalues of $\boldsymbol{Q}$ are positive (i.e., $\{\lambda_i > 0\}_{i=1}^n$).
2. $\boldsymbol{Q} \succ 0 \iff$ all leading principal minors of $\boldsymbol{Q}$ are positive. (Sylvester rule)

   hint of proof: by denoting Δ_i the i-order leading principal minors of $\boldsymbol{Q}$, there exists invertible matrix $\boldsymbol{V}$ such that

$$\boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} \xlongequal{\boldsymbol{x}=\boldsymbol{V}\tilde{\boldsymbol{x}}} \frac{\Delta_0}{\Delta_1}\tilde{x}_1^2 + \frac{\Delta_1}{\Delta_2}\tilde{x}_2^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n}\tilde{x}_n^2.$$

★ if $\boldsymbol{Q}$ is asymmetric, Sylvester rule cannot be used.

Example (counterexample)

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix},$$

Although $\Delta_1 = 1 > 0$ and $\Delta_2 = \det \boldsymbol{Q} = 1 > 0$, $\boldsymbol{Q} \not\succ 0$ ($\because \boldsymbol{x} = [1, 1]^\top \Longrightarrow \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} = -2 < 0$).

Inner Products and Norms

Theorem

$\boldsymbol{Q} \succeq 0 \Longrightarrow$ *all leading principal minors of* $\boldsymbol{Q}$ *are nonnegative.*

★ The above Theorem is not a sufficient condition.

Example (counterexample)

$$\boldsymbol{Q} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Although $\Delta_1 = 2$, $\Delta_2 = 0$ $\Delta_3 = 0$, $\boldsymbol{Q} \not\succeq 0$.
$(\because \boldsymbol{x} = [1, 1, -2]^\top \Leftrightarrow \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} < 0)$.

Theorem (positive semidefinite)

- $\boldsymbol{Q} \succeq 0 \iff$ *all principal minors of* $\boldsymbol{Q}$ *are nonnegative.*
- $\boldsymbol{Q} \succeq 0 \iff$ *all eigenvalues of* $\boldsymbol{Q}$ *are nonnegative.*

proof in linear algebra textbook.

Matrix Norms

Definition (matrix norm)

The norm of a matrix $\boldsymbol{A}$ is a function satisfying the following conditions:

1. $\|\boldsymbol{A}\| > 0$ if $\boldsymbol{A} \neq 0$, and $\|0\| = 0$, where 0 is a zero matrix.
2. $\|c\boldsymbol{A}\| = |c|\|\boldsymbol{A}\|$, $\forall c \in \mathbb{R}$.
3. $\|\boldsymbol{A}+\boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$ and $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$.

★ e.g., Frobenius norm (i.e., F-norm): $\|\boldsymbol{A}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$.

induced matrix norms

Let $\|\cdot\|_{(n)}$ and $\|\cdot\|_{(m)}$ be vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$. The induced matrix norm satisfies: $\|\boldsymbol{A}\boldsymbol{x}\|_{(m)} \leq \|\boldsymbol{A}\|\|\boldsymbol{x}\|_{(n)}$ for all $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{x} \in \mathbb{R}^n$, i.e.,

$$\|\boldsymbol{A}\| = \max_{\boldsymbol{x}\neq 0} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_{(m)}}{\|\boldsymbol{x}\|_{(n)}} = \max_{\|\boldsymbol{x}\|_{(n)}=1} \|\boldsymbol{A}\boldsymbol{x}\|_{(m)}.$$

Matrix Norms

Definition (Rayleigh's inequality)

Let $\lambda_{\max}(\boldsymbol{Q})$ and $\lambda_{\min}(\boldsymbol{Q})$ be the maximal and minimal eigenvalues of symmetric matrix $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$. Then, $\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{x}\|_{\boldsymbol{Q}}^2 \leq \lambda_{\max}(\boldsymbol{Q})\|\boldsymbol{x}\|_2^2$ for all $\boldsymbol{x} \neq 0$. (proof: quadratic form in linear algebra.)

Example (spectral norm: i.e., by setting $\|\cdot\|_{(n)}$ and $\|\cdot\|_{(m)}$ as 2-norm)

$\|\boldsymbol{A}\| = \sqrt{\lambda_1}$, where λ_1 is the largest eigenvalue of $\boldsymbol{A}^\top \boldsymbol{A}$.

proof. $\because$ $\boldsymbol{A}^\top \boldsymbol{A}$ is symmetric and positive semidefinite.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be its eigenvalues,
$\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be the corresponding orthonormal eigenvectors.

For any $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = 1$, it follows by Rayleight's inequality that

$\|\boldsymbol{A}\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} \rangle \leq \lambda_1 \|\boldsymbol{x}\|_2^2 = \lambda_1$.

For a unit eigenvector $\boldsymbol{x}$ of $\boldsymbol{A}^\top \boldsymbol{A}$ corresponding to the eigenvalue λ_1,

$\|\boldsymbol{A}\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{A}^\top \boldsymbol{A}\boldsymbol{x} \rangle = \lambda_1 \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \lambda_1$.

Matrix Norm

Example (compute the spectral norm of A)

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \ \|\boldsymbol{A}\| = ?$$

hint: $|\lambda I_2 - \boldsymbol{A}^\top \boldsymbol{A}| = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9) = 0,$
$\therefore \lambda_1 = 9, \therefore \|\boldsymbol{A}\| = \sqrt{\lambda_1} = 3.$

★ spectral radius and spectral norm.

- spectral radius: $\rho(\boldsymbol{A}) = \max\limits_{1 \le i \le n} |\lambda_i(\boldsymbol{A})|$. square matrix
- spectral norm: $\|\boldsymbol{A}\| = \max\limits_{1 \le i \le n} \sqrt{\lambda_i(\boldsymbol{A}^\top \boldsymbol{A})}$. all matrices.

★ Typically, $\rho(\boldsymbol{A}) \le \|\boldsymbol{A}\|$. "$=$" holds if $\boldsymbol{A}$ is symmetric. e.g.,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Longrightarrow \lambda_1(\boldsymbol{A}) = 0 \ \text{ and } \ \lambda_1(\boldsymbol{A}^\top \boldsymbol{A}) = 1.$$

Matrix Norms

Other induced matrix norms:

$$\|\boldsymbol{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|, \quad \|\boldsymbol{A}\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

Example (some operator norms $\|\cdot\|_{a,b}$ for matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$)

	$a=1$	$a=2$	$a=\infty$
$b=1$	$\max_{j=1,\ldots,n} \|\boldsymbol{A}_{:,j}\|_1$	$\max_{j=1,\ldots,n} \|\boldsymbol{A}_{:,j}\|_2$	$\max_{j=1,\ldots,n} \|\boldsymbol{A}_{:,j}\|_\infty$
$b=2$	NP-hard	$\left(\lambda_{\max}(\boldsymbol{A}^\top\boldsymbol{A})\right)^{1/2}$	$\max_{i=1,\ldots,m} \|\boldsymbol{A}_{i,:}\|_2$
$b=\infty$	NP-hard	NP-hard	$\max_{i=1,\ldots,m} \|\boldsymbol{A}_{i,:}\|_1$

★ Other norms (but not the induced matrix norm) for an $\boldsymbol{A} \in \mathbb{R}^{n\times n}$:

$$\|\boldsymbol{A}\|_* = \sum_{i=1}^{n} \sigma_i, \quad \|\boldsymbol{A}\|_F = \max_{1\le i\le n} \sigma_i, \quad \text{where } \boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^\top \text{ is SVD.}$$

Homework (Exercise in text book): 3.17, 3.21, 3.22