Chapter 24 Multi-objective Optimization

1. Pareto Solutions
2. Computing the Pareto Front
3. From Multi-objective to Single-objective Optimization

Multi-objective Optimization

- single-objective optimization: involves only one objective function;
- multi-objective (multicriteria/vector) optimization: involves a number of conflicting objectives. The objectives are in conflict with each other if an improvement in one objective leads to deterioration in another.

Definition (multi-objective optimization problem)

$$\min \ \boldsymbol{f}(\boldsymbol{x}), \ \text{s.t.} \ \boldsymbol{x} \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^l$, i.e., $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_\ell(x_1, x_2, \ldots, x_n) \end{bmatrix}$.

★ (1) may have no unique optimal solution.

purpose: find $\boldsymbol{x}^* \in \Omega$ and optimizes $\boldsymbol{f}$.

three different types of multi-objective optimization problems:

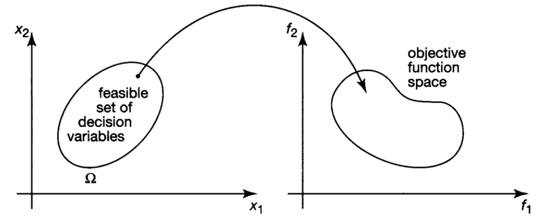
1. Minimize all the objective functions.
2. Maximize all the objective functions.
3. Minimize some and maximize others.

★ ❷ and ❸ can be converted into ❶.

Example (decision variable space, objective function space)

$$\min \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix},$$

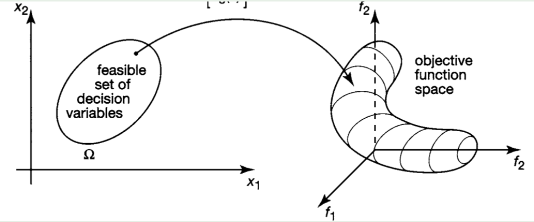
s.t. $\boldsymbol{x} \in \mathbb{R}^2$.



Example (decision variable space, objective function space)

$$\min \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix},$$

s.t. $\boldsymbol{x} \in \mathbb{R}^2$.



- ★ difference between single/multi-objective optimization: The former focuses on the decision variable space; the latter focuses on the objective space.
- ★ difficulty of multi-objective optimization: No natural ordering in the objective space.

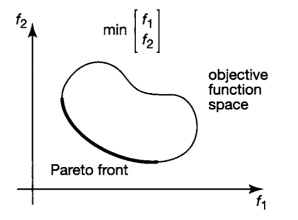
Pareto solutions of multi-objective optimization (1)

For an $\boldsymbol{x}^* \in \Omega$, if $\nexists \boldsymbol{x} \in \Omega$ such that for $i = 1, \ldots, \ell$, $f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{x}^*)$, and for at least one i, $f_i(\boldsymbol{x}) < f_i(\boldsymbol{x}^*)$, then $\boldsymbol{x}^*$ is called a Pareto minimizer (nondominated solution); the set of Pareto minimizers (optimizers) is called the Pareto front.

Remark: $\boldsymbol{x}^*$ is a Pareto minimizer

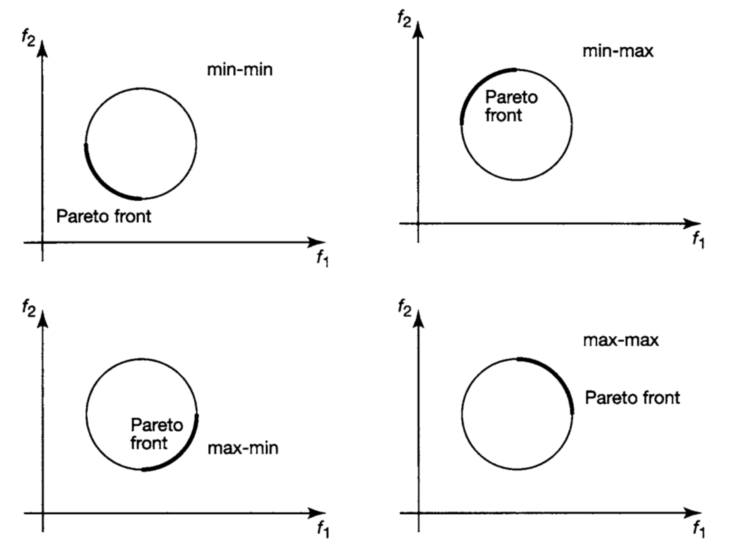
$$\Downarrow$$

$\nexists$ other feasible decision variable $\boldsymbol{x}$ that would decrease some objectives without causing simultaneous increase in at least one other variable.



Pareto Solutions

Example (Pareto fronts of two-objective optimization)



computing the pareto front: two solutions are compared and the dominated solution is eliminated from the set of candidates of Pareto optimizers.

Notation: the rth candidate Pareto optimal solution:

$$\boldsymbol{x}^{*r} = \left[x_1^{*r}, x_2^{*r}, \ldots, x_n^{*r}\right]^\top, \; r = 1, 2, ..., R,$$

where R is the number of current candidate Pareto solutions.

objective function value at point $\boldsymbol{x}^{*r}$:

$$\boldsymbol{f}\left(\boldsymbol{x}^{*r}\right) = \left[f_1\left(\boldsymbol{x}^{*r}\right), f_2\left(\boldsymbol{x}^{*r}\right), \ldots, f_\ell\left(\boldsymbol{x}^{*r}\right)\right]^\top.$$

Pareto Solutions

Comparison: for any new solution candidate $\boldsymbol{x}^j$, when evaluate the objective function vector $\boldsymbol{f}(\boldsymbol{x}^j)$, there are three cases:

1. $\boldsymbol{x}^j$ dominates at least one candidate solution.
2. $\boldsymbol{x}^j$ does not dominate any existing candidate solutions.
3. $\boldsymbol{x}^j$ is dominated by a candidate solution.

★ for ❶, delete the dominated solutions from the set and add the new solution $\boldsymbol{x}^j$ to the set of candidates.

★ for ❷, add this new Pareto solution to the set of candidate Pareto solutions.

★ for ❸, do not change the set of the existing candidate Pareto solutions.

Pareto Solutions

Example (two-objective minimization problem)

$\boldsymbol{x}^*$ is a nondominated point, then $\forall i, \boldsymbol{x}$, $f_i(\boldsymbol{x}^*) \leq f_i(\boldsymbol{x})$, and at least for one component j of the objective vector, $f_j(\boldsymbol{x}^*) < f_j(\boldsymbol{x})$.

$\boldsymbol{x}^{(i)\top}$	$\boldsymbol{f}(\boldsymbol{x}^{(i)})^\top$
$[5,6]$	$[30,45]$
$[4,5]$	$[22,29]$
$[3,7]$	$[19,53]$
$[6,8]$	$[41,75]$
$[1,4]$	$[13,45]$
$[6,7]$	$[42,55]$
$[2,5]$	$[37,46]$
$[3,6]$	$[28,37]$
$[2,7]$	$[12,51]$
$[4,7]$	$[41,67]$

Sol: start with the first pair as a candidate Pareto optimal solution and then compare the other pairs against this first pair, replacing the first pair as necessary. Then, continue with the other pairs. The result of the search gives the following Pareto optimal set:

$\boldsymbol{x}^{(i)\top}$	$\boldsymbol{f}(\boldsymbol{x}^{(i)})^\top$
$[4,5]$	$[22,29]$
$[1,4]$	$[13,45]$
$[2,7]$	$[12,51]$

Algorithm for Generating a Pareto Front

Notation:

J: the number of candidate solutions to be checked for optimality.
R: the number of current candidate Pareto solutions.
ℓ: the number of objective functions, i.e. the dimension of the objective function vector.
n: the dimension of the decision space, i.e. the number of components of $\boldsymbol{x}$.

Algorithm

1. Generate an initial solution $\boldsymbol{x}^1$ and evaluate $\boldsymbol{f}^{*1} = \boldsymbol{f}\left(\boldsymbol{x}^1\right)$. $\boldsymbol{x}^1$ is taken as a candidate Pareto solution. Set initial indices $R := 1$ and $j := 1$.
2. Set $j := j + 1$. If $j \leq J$, then generate solution $\boldsymbol{x}^j$ and go to step 3 . Otherwise, stop.

Algorithm for Generating a Pareto Front

Algorithm

3. Set $r := 1$ and $q := 0$ (q represents the number of eliminated solutions from the existing set of Pareto solutions).
4. If $\forall i = 1, 2, \ldots, \ell,\ f_i\left(\boldsymbol{x}^j\right) < f_i\left(\boldsymbol{x}^{*r}\right)$, then set $q := q + 1$, $\boldsymbol{f}^{*R} := \boldsymbol{f}\left(\boldsymbol{x}^j\right)$, remember $\boldsymbol{x}^{*r}$ that should be eliminated, and go to step 6.
5. If $\forall i = 1, 2, \ldots, \ell,\ f_i\left(\boldsymbol{x}^j\right) \geq f_i\left(\boldsymbol{x}^{*r}\right)$, then go to step 2.
6. Set $r := r + 1$. If $r \leq R$, go to step 4.
7. If $q \neq 0$, remove from the candidate Pareto set the solutions that are eliminated in step 4, add solution $\boldsymbol{x}^j$ as a new candidate Pareto solution, and go to step 2.
8. Set $R := R + 1, \boldsymbol{x}^{*R} := \boldsymbol{x}^j, \boldsymbol{f}^{*R} := \boldsymbol{f}\left(\boldsymbol{x}^j\right)$, and go to step 2.

From Multi-objective to Single-objective Optimization

★ discuss four techniques to convert a multi-objective problem to a single-objective problem.

method I: weighted-sum method

To form a single-objective function by taking a convex combination of the components of the objective function vector, i.e.

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{f}(\boldsymbol{x}),$$

$\boldsymbol{c}$ is a vector of positive components.

Trouble: it might be difficult to determine suitable weight ($\boldsymbol{c}$) values.

method II: minimax method

To form a single-objective function by taking the maximum of the components of the objective vector, i.e.

$$\boldsymbol{f}(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), \cdots, f_\ell(\boldsymbol{x})\}.$$

Trouble: $f_1, \cdots, f_\ell$ are in the same units (i.e., the unit is consistent); $\boldsymbol{f}$ might not be differentiable.

From Multi-objective to Single-objective Optimization

★ a minimax problem with linear objective vector components and linear constraints can be reduced to a linear program.

Example (Given vectors $\boldsymbol{v_1}, \cdots, \boldsymbol{v_p} \in \mathbb{R}^n$ and scalars $u_1, \cdots, u_p$)

(I)
$$\min \max\{\boldsymbol{v}_1^\top \boldsymbol{x} + u_1, \cdots, \boldsymbol{v}_p^\top \boldsymbol{x} + u_p\},$$
$$\text{s.t. } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},$$
where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$.

(II)
$$\begin{aligned} \min \quad & y, \\ \text{s.t. } \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \\ & y \ge \boldsymbol{v}_i^\top \boldsymbol{x} + u_i, \\ & i = 1, \cdots, p. \end{aligned}$$

Show that $\boldsymbol{x}^*$ solves (I) $\Leftrightarrow [\boldsymbol{x}^{*\top}, y^*]^\top$ solves (II) with $y^* = \max\{\boldsymbol{v}_1^\top \boldsymbol{x}^* + u_1, \cdots, \boldsymbol{v}_p^\top \boldsymbol{x}^* + u_p\}$.

proof: $\Rightarrow$) Suppose that $\boldsymbol{x}^*$ is optimal in (I).

Let $y^* = \max\left\{\boldsymbol{v}_1^\top \boldsymbol{x}^* + u_1, \ldots, \boldsymbol{v}_p^\top \boldsymbol{x}^* + u_p\right\}$,

$\therefore \left[\boldsymbol{x}^{*\top}, y^*\right]^\top$ is feasible in (II).

Let $\left[\boldsymbol{x}^\top, y\right]^\top$ be any feasible point in (II).

$\because$ $\boldsymbol{x}$ is feasible in (I)

$$\therefore\ y \geq \max\left\{\boldsymbol{v}_1^\top \boldsymbol{x} + u_1, \ldots, \boldsymbol{v}_p^\top \boldsymbol{x} + u_p\right\}$$
$$\geq \max\left\{\boldsymbol{v}_1^\top \boldsymbol{x}^* + u_1, \ldots, \boldsymbol{v}_p^\top \boldsymbol{x}^* + u_p\right\} = y^*.$$

$\therefore$ $\left[\boldsymbol{x}^{*\top}, y^*\right]^\top$ solves (II).

$\Leftarrow$) Suppose that $\boldsymbol{x}^*$ is not optimal in (I).

$\therefore$ $\exists\, \boldsymbol{x}'$ is feasible in (I) such that

$$y' = \max\left\{\boldsymbol{v}_1^\top \boldsymbol{x}' + u_1, \cdots, \boldsymbol{v}_p^\top \boldsymbol{x}' + u_p\right\}$$
$$< \max\left\{\boldsymbol{v}_1^\top \boldsymbol{x}^* + u_1, \cdots, \boldsymbol{v}_p^\top \boldsymbol{x}^* + u_p\right\} = y^*.$$

$\because$ $\left[\boldsymbol{x}'^\top, y'\right]^\top$ is evidently feasible in (II), and has objective function value (y') that is lower than that of $\left[\boldsymbol{x}^{*\top}, y^*\right]^\top$.

$\therefore$ $\left[\boldsymbol{x}^{*\top}, y^*\right]^\top$ is not optimal in (II).

From Multi-objective to Single-objective Optimization

method III

Assuming that the components of the objective vector are nonnegative, to form a single-objective function by taking the p-norm of the objective vector, i.e.

$$f(\boldsymbol{x}) = \|\boldsymbol{f}(\boldsymbol{x})\|_p \text{ or } f(\boldsymbol{x}) = \|\boldsymbol{f}(\boldsymbol{x})\|_p^p = (f_1(\boldsymbol{x}))^p + \cdots + (f_\ell(\boldsymbol{x}))^p.$$

★ the second method can be viewed as a special case of this method, with $p = \infty$.

The fourth method

To minimize one of the components of the objective vector subject to constraints on the other components.

$$\begin{aligned} &\min f_1(\boldsymbol{x}),\\ \text{s.t. } &f_2(\boldsymbol{x}) \le b_2,\\ &\quad\vdots\\ &f_\ell(\boldsymbol{x}) \le b_\ell; \end{aligned}$$

where $b_2, \cdots, b_\ell$ are given constants that reflect satisfactory values for the objectives $f_2, \cdots, f_\ell$.

Remark: this approach is suitable only in situations where these satisfactory values can be determined.