# Chapter 17 Dual Simplex Method

1. Dual Linear Program
2. Dual Simplex Method

# Dual Linear Program

## primal and dual of LP

primal problem (P)

$$\boxed{\begin{aligned} &\min\ \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{s.t.}\ \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \ \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}}$$

$$\xRightarrow[\text{writing the dual problem}]{\text{why?}}$$

# Dual Linear Program

## primal and dual of LP

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$$\text{s.t.} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$$

$$\xrightarrow[\text{writing the dual problem}]{\text{why?}}$$

**dual problem (D)**

$$\min \ \boldsymbol{\lambda}^\top \boldsymbol{b}$$
$$\text{s.t.} \ \boldsymbol{A}^\top \boldsymbol{\lambda} \leq \boldsymbol{c}.$$

★ Other forms of dual problems can be derived from the principle of duality.

## Example (write the dual of the following LP)

$$\begin{array}{rl}
\min & 8x_1 + 6x_2 + 3x_3 + 6x_4 \\
\text{s.t.} & x_1 + 2x_2 \qquad + x_4 \geq 3 \\
& 3x_1 + x_2 + x_3 + x_4 \geq 6 \\
& \qquad\qquad x_3 + x_4 \geq 2 \\
& x_j \geq 0, \ j = 1, \cdots, 4.
\end{array}$$

# Dual Linear Program

## primal and dual of LP

**primal problem (P)**

$$\min \ \boldsymbol{c}^\top \boldsymbol{x}$$
$$\text{s.t.} \ \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$$

$$\xrightarrow[\text{writing the dual problem}]{\text{why?}}$$

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\min & 8x_1 + 6x_2 + 3x_3 + 6x_4 \\
\text{s.t.} & x_1 + 2x_2 \phantom{+x_3} + x_4 \geq 3 \\
 & 3x_1 + x_2 + x_3 + x_4 \geq 6 \\
 & x_3 + x_4 \geq 2 \\
 & x_j \geq 0, \ j = 1, \cdots, 4.
\end{array}
\Longrightarrow
\begin{array}{rl}
\min & 3w_1 + 6w_2 + 2w_3 \\
\text{s.t.} & w_1 + 3w_2 \leq 8 \\
 & 2w_1 + w_2 \leq 6 \\
 & w_2 + w_3 \leq 3 \\
 & w_1 + w_2 + w_3 \leq 6 \\
 & w_j \geq 0, \ j = 1, 2, 3.
\end{array}$$

# Dual Linear Program

## Lemma (weak duality)

*Suppose that* $\boldsymbol{x}$ *and* $\boldsymbol{\lambda}$ *are feasible solutions to primal and dual LP problems, respectively. Then,* $\boldsymbol{c}^\top \boldsymbol{x} \geq \boldsymbol{\lambda}^\top \boldsymbol{b}$*.*
*(Proof: On the blackboard.)*

## Theorem

*Suppose that* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are feasible solutions to the primal and dual, respectively. If* $\boldsymbol{c}^\top \boldsymbol{x}_0 = \boldsymbol{\lambda}_0^\top \boldsymbol{b}$*, then* $\boldsymbol{x}_0$ *and* $\boldsymbol{\lambda}_0$ *are optimal feasible solutions to their respective problems.*
*(Proof: On the blackboard.)*

**Theorem (duality theorem)**

*If primal problem has an optimal solution, then so does the dual. Moreover, the optimal values of their respective objective functions are equal.*

proof. Case 1. assume that the dual problem has a feasible solution $\boldsymbol{\lambda}$.

Analyze the initial and final simplex of standard LP

initial simplex of standard LP final simplex of standard LP

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^\top & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^\top & \boldsymbol{c}_D^\top & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^\top & \boldsymbol{c}_D^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}$$

We can observe that:

1. $\boldsymbol{r}_D^\top = \boldsymbol{c}_D^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} \geq 0 \Longrightarrow \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} \leq \boldsymbol{c}_D^\top$
   By denoting $\boldsymbol{\lambda}^\top := \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}$, then $\boldsymbol{\lambda}^\top \boldsymbol{D} \leq \boldsymbol{c}_D^\top$.
   $\because \boldsymbol{\lambda}^\top \boldsymbol{A} = \boldsymbol{\lambda}^\top [\boldsymbol{B}, \boldsymbol{D}] = [\boldsymbol{\lambda}^\top \boldsymbol{B}, \boldsymbol{\lambda}^\top \boldsymbol{D}] = [\boldsymbol{c}_B^\top, \boldsymbol{\lambda}^\top \boldsymbol{D}] \leq [\boldsymbol{c}_B^\top, \boldsymbol{c}_D^\top] = \boldsymbol{c}^\top$,
   $\therefore \boldsymbol{A}^\top \boldsymbol{\lambda} \leq \boldsymbol{c}$, which implies $\boldsymbol{\lambda} = \boldsymbol{B}^{-\top} \boldsymbol{c}_B$ is a f.s. of the dual LP.

# Dual Linear Program

2. we now prove $\boldsymbol{\lambda}$ is an optimal f.s. of the dual LP.
   $\because \boldsymbol{\lambda}^\top \boldsymbol{b} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b}$ =optimal function value,
   $\therefore \boldsymbol{\lambda}$ is a optimum of the dual LP.

## how to get the dual solution from the final simplex?

initial simplex of standard LP　　final simplex of standard LP

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^\top & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^\top & \boldsymbol{c}_D^\top & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boldsymbol{I} & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^\top & \boldsymbol{c}_D^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}$$

- case 1: rank$(\boldsymbol{D}) = m$. Solving linear equations

$$\boldsymbol{\lambda}^\top \boldsymbol{D} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} = \boldsymbol{c}_D^\top - \boldsymbol{r}_D^\top.$$

- case 2: rank$(\boldsymbol{D}) < m$. Solving linear equations

$$\begin{cases} \boldsymbol{\lambda}^\top \boldsymbol{B} = \boldsymbol{c}_B^\top \\ \boldsymbol{\lambda}^\top \boldsymbol{D} = \boldsymbol{c}_B^\top - \boldsymbol{r}_D^\top \end{cases} \Longleftrightarrow \boldsymbol{\lambda}^\top \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_B^\top \\ \boldsymbol{c}_B^\top - \boldsymbol{r}_D^\top \end{bmatrix} \Longleftrightarrow \boldsymbol{\lambda}^\top \boldsymbol{A} = \boldsymbol{c}^\top - \boldsymbol{r}^\top$$

## Example (write the dual of LP and find the optimum)

$$
\begin{array}{rl}
\max & 2x_1 + 5x_2 + x_3 \\
\text{s.t.} & 2x_1 \quad -x_2 + 7x_3 \le 6 \\
 & \ \ x_1 + 3x_2 + 4x_3 \le 9 \\
 & 3x_1 + 6x_2 \quad + x_3 \le 3 \\
 & x_i \ge 0,\ i = 1, 2, 3
\end{array}
$$

# Dual Linear Program

## Example (write the dual of LP and find the optimum)

$$
\begin{array}{rl}
\max & 2x_1 + 5x_2 + x_3 \\
\text{s.t.} & 2x_1 - x_2 + 7x_3 \le 6 \\
& x_1 + 3x_2 + 4x_3 \le 9 \\
& 3x_1 + 6x_2 + x_3 \le 3 \\
& x_i \ge 0,\ i = 1,2,3
\end{array}
\xrightarrow[\text{form}]{\text{standard}}
\begin{array}{rl}
\min & -2x_1 - 5x_2 - x_3 \\
\text{s.t.} & 2x_1 - x_2 + 7x_3 + x_4 = 6 \\
& x_1 + 3x_2 + 4x_3 + x_5 = 9 \\
& 3x_1 + 6x_2 + x_3 + x_6 = 3 \\
& x_i \ge 0,\ i = 1,\cdots,6
\end{array}
$$

initial simplex

| $\boldsymbol{a}_1$ | $\boldsymbol{a}_2$ | $\boldsymbol{a}_3$ | $\boldsymbol{a}_4$ | $\boldsymbol{a}_5$ | $\boldsymbol{a}_6$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|
| 2 | $-1$ | 7 | 1 | 0 | 0 | 6 |
| 1 | 3 | 4 | 0 | 1 | 0 | 9 |
| 3 | 6 | 1 | 0 | 0 | 1 | 3 |
| $-2$ | $-5$ | $-1$ | 0 | 0 | 0 | 0 |

$\Rightarrow \cdots \Rightarrow \cdots \Rightarrow$

final simplex

| $\boldsymbol{a}_1$ | $\boldsymbol{a}_2$ | $\boldsymbol{a}_3$ | $\boldsymbol{a}_4$ | $\boldsymbol{a}_5$ | $\boldsymbol{a}_6$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|
| $\frac{15}{43}$ | 0 | 1 | $\frac{6}{43}$ | 0 | $\frac{1}{43}$ | $\frac{39}{43}$ |
| $-\frac{74}{43}$ | 0 | 0 | $-\frac{21}{43}$ | 1 | $-\frac{25}{43}$ | $\frac{186}{43}$ |
| $\frac{19}{43}$ | 1 | 0 | $-\frac{1}{43}$ | 0 | $\frac{7}{43}$ | $\frac{15}{43}$ |
| $\frac{24}{43}$ | 0 | 0 | $\frac{1}{43}$ | 0 | $\frac{36}{43}$ | $\frac{114}{43}$ |

$$\therefore \boldsymbol{\lambda}^\top \boldsymbol{D} = \boldsymbol{c}_{\boldsymbol{D}}^\top - \boldsymbol{r}_{\boldsymbol{D}}^\top \Rightarrow [\lambda_1, \lambda_2, \lambda_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix} = [-2, 0, 0] - \left[\frac{24}{43}, \frac{1}{43}, \frac{36}{43}\right]$$

$$\therefore \boldsymbol{\lambda}^\top = \left[-\frac{1}{43}, 0, -\frac{36}{43}\right]$$

# Dual Linear Program

**Theorem (complementary slackness)**

*The feasible solutions pairs* $(\boldsymbol{x}, \boldsymbol{\lambda})$ *are optimal for primal and dual LPs* $\Longleftrightarrow$ $(\boldsymbol{c}^\top - \boldsymbol{\lambda}^\top \boldsymbol{A})\, \boldsymbol{x} = 0$ *and* $\boldsymbol{\lambda}^\top (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0$.

proof. Use complementary relaxation theory for constrained optimization.

**Example (write the dual of LP and find the optimum)**

$$\begin{array}{ll} \min & -(2x_1 + x_2 + 7x_3 + 4x_4) \\ \text{s.t.} & x_1 + x_2 + x_3 \quad + x_4 = 26 \\ & x_i \geq 0,\ i = 1, \cdots, 4 \end{array} \xrightarrow{\text{dual}} \begin{array}{ll} \max & 26\lambda \\ \text{s.t.} & \lambda \leq -2,\ \lambda \leq -1,\ \lambda \leq -7,\ \lambda \leq -4 \end{array}$$

Obviously, the maximizer of dual is $\lambda = -7$.

How to get minimizer of primal by complementary slackness?

Ans: primal minimizer $\boldsymbol{x}$ of satisfies

$$\begin{cases} (\boldsymbol{c}^\top - \boldsymbol{\lambda}^\top \boldsymbol{A})\, \boldsymbol{x} = 0 \\ \boldsymbol{\lambda}^\top (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) = 0 \end{cases} \xrightarrow{\text{i.e.,}} \begin{cases} (-[2, 1, 7, 4] - (-7)[1, 1, 1, 1])\boldsymbol{x} = 0 \\ [1, 1, 1, 1]\boldsymbol{x} - 26 = 0 \end{cases}$$

$\therefore \boldsymbol{x} = [0, 0, 26, 0]^\top$.

# Dual Linear Program

## Example (write the dual of LP and find the optimum)

$$\begin{array}{ll} \max & 2x_1 + 5x_2 + x_3 \\ \text{s.t.} & 2x_1 - x_2 + 7x_3 \le 6 \\ & x_1 + 3x_2 + 4x_3 \le 9 \\ & 3x_1 + 6x_2 + x_3 \le 3 \\ & x_i \ge 0,\ i = 1, \cdots, 3 \end{array} \xRightarrow{\text{dual}} \begin{array}{ll} \min & -2x_1 - 5x_2 - x_3 \\ \text{s.t.} & 2x_1 - x_2 + 7x_3 + x_4 = 6 \\ & x_1 + 3x_2 + 4x_3 + x_5 = 9 \\ & 3x_1 + 6x_2 + x_3 + x_6 = 3 \\ & x_i \ge 0,\ i = 1, \cdots, 6 \end{array}$$

$$\because \boxed{\begin{array}{ll} \min & [\boldsymbol{c}^\top, \boldsymbol{0}^\top]\boldsymbol{x} \\ \text{s.t.} & [\boldsymbol{A}, \boldsymbol{I}]\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{array}} \xRightarrow{\text{dual}} \boxed{\begin{array}{ll} \max & \boldsymbol{\lambda}^\top \boldsymbol{b} \\ \text{s.t.} & \boldsymbol{\lambda}^\top [\boldsymbol{A}, \boldsymbol{I}] \le [\boldsymbol{c}^\top, \boldsymbol{0}^\top] \\ & \boldsymbol{\lambda} \ge \boldsymbol{0} \end{array}}$$

By defining $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}$.

Use the two-phase method to solve the dual LP.

Motivation: For a standard LP, when $\boldsymbol{A}$ has no identity matrix, we should use two-phase method or big $M$ method.

Tip: solve a standard LP by using the properties its dual, find the optima without introducing artificial variables ($\boldsymbol{b} \geq 0$ can be unnecessary).

# Dual Simplex Method

standard LP
$$\min \quad \boldsymbol{c}^\top \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}.$$
$\Longrightarrow$

$$\begin{array}{ccccccccc|c}
x_1 & & & & x_{m+1} & \cdots & x_l & \cdots & x_n & \boldsymbol{b} \\
1 & & & & a_{1,m+1} & \cdots & a_{1l} & \cdots & a_{1n} & b_1 \\
& \ddots & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & 1 & & a_{k,m+1} & \cdots & \boxed{a_{kl}} & \cdots & a_{kn} & b_k \\
& & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & 1 & a_{m,m+1} & \cdots & a_{ml} & \cdots & a_{mn} & b_m \\
\hline
0 & \cdots\ 0 & \cdots & 0 & \sigma_{m+1} & \cdots & \sigma_l & \cdots & \sigma_n & f_0
\end{array}$$

If $b_k < 0$ and update the simplex with pivot element $a_{kl}$, then we have

$$b'_k = \frac{b_k}{a_{kl}} \text{ and } \sigma'_j = \sigma_j - \frac{a_{kj}}{a_{kl}}\sigma_l, j = m+1, \cdots, n. \tag{1}$$

Thus, the pivot element $a_{kl}$ should satisfy

$$a_{kl} < 0, \quad \text{and} \quad \frac{\sigma_l}{a_{kl}} = \max\left\{\frac{\sigma_j}{a_{kj}} \mid a_{kj} < 0\right\}. \tag{2}$$

## Example (solve LP by dual simplex method)

original LP

$$
\begin{array}{lrrl}
\min & x_1 + 2x_2 & & \\
\text{s.t.} & x_1 & +2x_2 & \geq 4 \\
 & x_1 & & \leq 5 \\
 & 3x_1 & +x_2 & \geq 6 \\
 & & x_i \geq 0, & i = 1, 2
\end{array}
$$

## Example (solve LP by dual simplex method)

**original LP**

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 4 \\ & x_1 \leq 5 \\ & 3x_1 + x_2 \geq 6 \\ & x_i \geq 0,\ i = 1, 2 \end{array}$$

$\xrightarrow[\text{form}]{\text{stardard}}$

**standard form LP (P0)**

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 - x_3 = 4 \\ & x_1 + x_4 = 5 \\ & 3x_1 + x_2 - x_5 = 6 \\ & x_i \geq 0,\ i = 1, \cdots, 5 \end{array}$$

Note: the above LP can not be solved by simplex, should be applied two-phase method or big $M$ method. Now, we solve it by dual simplex method.

Ans:

matrix form of (P0)

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|
| 1 | 2 | $-1$ | 0 | 0 | 4 |
| 1 | 0 | 0 | 1 | 0 | 5 |
| 3 | 1 | 0 | 0 | $-1$ | 6 |
| 1 | 2 | 0 | 0 | 0 | 0 |

# Dual Simplex Method

## Example (solve LP by dual simplex method)

**original LP**

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 \geq 4 \\ & x_1 \leq 5 \\ & 3x_1 + x_2 \geq 6 \\ & x_i \geq 0,\ i = 1,2 \end{array}$$

$\xrightarrow[\text{form}]{\text{stardard}}$

**standard form LP (P0)**

$$\begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + 2x_2 - x_3 = 4 \\ & x_1 + x_4 = 5 \\ & 3x_1 + x_2 - x_5 = 6 \\ & x_i \geq 0,\ i = 1, \cdots, 5 \end{array}$$

Note: the above LP can not be solved by simplex, should be applied two-phase method or big $M$ method. Now, we solve it by dual simplex method.

Ans:

matrix form of (P0)

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|
| 1 | 2 | $-1$ | 0 | 0 | 4 |
| 1 | 0 | 0 | 1 | 0 | 5 |
| 3 | 1 | 0 | 0 | $-1$ | 6 |
| 1 | 2 | 0 | 0 | 0 | 0 |

$\Rightarrow$ scaling 1st and 3rd rows

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|
| $\boxed{-1}$ | $-2$ | 1 | 0 | 0 | $-4$ |
| 1 | 0 | 0 | 1 | 0 | 5 |
| $-3$ | $-1$ | 0 | 0 | 1 | $-6$ |
| 1 | 2 | 0 | 0 | 0 | 0 |

$\because$ $\boldsymbol{r_D} \geq 0$ and some $\boldsymbol{b} < 0$

$\therefore$ operate on $b_i < 0$.

e.g., operate 1st row.

$$\Longrightarrow \begin{array}{|cccccc|}\hline x_1 & x_2 & x_3 & x_4 & x_5 & \boldsymbol{b} \\ 1 & 2 & -1 & 0 & 0 & 4 \\ 0 & -2 & 1 & 1 & 0 & 1 \\ 0 & 5 & -3 & 0 & 1 & 6 \\ \hline 0 & 0 & 1 & 0 & 0 & -4 \\ \hline \end{array}$$

$\because$ $\boldsymbol{r_D} \geq 0$ and $\boldsymbol{b} \geq 0$,
$\therefore$ (P0) is solved.
optimal f.s is $(x_1, x_2) = (4, 0)$.

## Example (solve LP by dual simplex method)

**original LP**

$$\begin{array}{ll} \min & 12x_1 + 8x_2 + 16x_3 + 12x_4 \\ \text{s.t.} & 2x_1 + x_2 + 4x_3 \geq 2 \\ & 2x_1 + 2x_2 + 4x_4 \geq 3 \\ & x_i \geq 0, \ i = 1, \cdots, 4 \end{array}$$

$\Rightarrow$

**standard form LP (P0)**

$$\begin{array}{ll} \min & 12x_1 + 8x_2 + 16x_3 + 12x_4 \\ \text{s.t.} & 2x_1 + x_2 + 4x_3 - x_5 = 2 \\ & 2x_1 + 2x_2 + 4x_4 - x_6 = 3 \\ & x_i \geq 0, \ i = 1, \cdots, 6 \end{array}$$

# Dual Simplex Method

matrix form of (P0)

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|
| $2$ | $1$ | $4$ | $0$ | $-1$ | $0$ | $2$ |
| $2$ | $2$ | $0$ | $4$ | $0$ | $-1$ | $3$ |
| $12$ | $8$ | $16$ | $12$ | $0$ | $0$ | $0$ |

scaling 1st and 2nd rows

$\Rightarrow$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|
| $-2$ | $-1$ | $-4$ | $0$ | $1$ | $0$ | $-2$ |
| $-2$ | $-2$ | $0$ | $\boxed{-4}$ | $0$ | $1$ | $-3$ |
| $12$ | $8$ | $16$ | $12$ | $0$ | $0$ | $0$ |

$\because$ $\boldsymbol{r_D} \geq 0$ and some $\boldsymbol{b} < 0$

$\therefore$ operate on $b_i < 0$.

e.g., operate 2nd row.

$\Rightarrow$

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|
| $-2$ | $\boxed{-1}$ | $-4$ | $0$ | $1$ | $0$ | $-2$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $1$ | $0$ | $-\frac{1}{4}$ | $\frac{3}{4}$ |
| $6$ | $2$ | $16$ | $0$ | $0$ | $3$ | $-9$ |

$\because$ $\boldsymbol{r_D} \geq 0$ and some $\boldsymbol{b} < 0$

$\therefore$ operate on $b_i < 0$.

e.g., operate 1st row.

$$\Rightarrow \begin{array}{|cccccc|c|} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \boldsymbol{b} \\ 2 & 1 & 4 & 0 & -1 & 0 & 2 \\ \boxed{-\frac{1}{2}} & 0 & -2 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \hline 2 & 0 & 8 & 0 & 2 & 3 & -13 \\ \hline \end{array}$$

$\because$ $\boldsymbol{r_D} \geq 0$ and some $\boldsymbol{b} < 0$
$\therefore$ operate on $b_i < 0$.
e.g., operate 2nd row.

$$\Rightarrow \begin{array}{|cccccc|c|} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \boldsymbol{b} \\ 0 & 1 & -4 & 4 & 1 & -1 & 1 \\ 1 & 0 & 4 & -2 & -1 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & 4 & 4 & 2 & -14 \\ \hline \end{array}$$

$\because$ $\boldsymbol{r_D} \geq 0$ and some $\boldsymbol{b} < 0$
$\therefore$ operate on $b_i < 0$.
e.g., operate 2nd row.

Thus, the optimal solution is $(x_1, x_2, x_3, x_4) = (1/2, 1, 0, 0)$.

Note: We can also get the optimal solution of the dual as $(\lambda_1, \lambda_2) = (4, 2)$.

# Homework

Exercise in the text book: 17.5,17.10,17.15,17.24.