Chapter 16 Simplex Method

Elementary Row Operations

revisit: elementary row operations

elementary row operation on matrix is an algebraic manipulation corresponds to one of the followings:

1. interchanging any two rows of the matrix
2. multiplying one of its rows by a real nonzero number
3. adding a scalar multiple of one row to another row.

linear equation can be transformed into canonical form by a finite number elementary row operations: $\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \text{with } \operatorname{rank}(\boldsymbol{A}) = m}$

$$\Longleftrightarrow \boxed{\begin{array}{llllll} x_1 & & & +a_{1m+1}x_{m+1} + \cdots + a_{1n}x_n & = b_1 \\ & x_2 & & +a_{2m+1}x_{m+1} + \cdots + a_{2n}x_n & = b_2 \\ & & \ddots & \vdots & \\ & & x_m & +a_{mm+1}x_{m+1} + \cdots + a_{mn}x_n & = b_m \end{array}}$$

$$\Longrightarrow \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{D}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ 0 \end{bmatrix} \text{ is a b.f.s.}$$

★ $\boldsymbol{x}_{\boldsymbol{B}}^{\top}$ may be not $(x_1, \cdots, x_m)$, the above canonical form is a special case.

Elementary Row Operations

How to obtain another b.f.s of $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ by changing the canonical form?

$$\begin{bmatrix}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_k & \ldots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \ldots & \boldsymbol{a}_l & \ldots & \boldsymbol{a}_n & \boldsymbol{b} \\
1 & 0 & \ldots & 0 & \ldots & 0 & a_{1,m+1} & \ldots & a_{1l} & \ldots & a_{1n} & b_1 \\
0 & 1 & \ldots & 0 & \ldots & 0 & a_{2,m+1} & \ldots & a_{2l} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & & 0 & a_{k,m+1} & \ldots & \boxed{a_{kl}} & \ldots & a_{kn} & b_k \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 1 & a_{m,m+1} & \ldots & a_{ml} & \ldots & a_{mn} & b_m
\end{bmatrix}$$

Elementary Row Operations

How to obtain another b.f.s of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ by changing the canonical form?

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_k & \ldots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \ldots & \boldsymbol{a}_l & \ldots & \boldsymbol{a}_n & \boldsymbol{b}
\end{array} \\
\begin{bmatrix}
1 & 0 & \ldots & 0 & \ldots & 0 & a_{1,m+1} & \ldots & a_{1l} & \ldots & a_{1n} & b_1 \\
0 & 1 & \ldots & 0 & \ldots & 0 & a_{2,m+1} & \ldots & a_{2l} & \ldots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & & 0 & a_{k,m+1} & \ldots & \boxed{a_{kl}} & \ldots & a_{kn} & b_k \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & 1 & a_{m,m+1} & \ldots & a_{ml} & \ldots & a_{mn} & b_m
\end{bmatrix}
\end{array}
\xRightarrow[\text{solution}]{\text{basic}}
\begin{bmatrix}
x_1 = b_1 \\
x_2 = b_2 \\
\vdots \\
x_m = b_m \\
x_{m+1} = 0 \\
\vdots \\
x_n = 0
\end{bmatrix}
$$

$\Downarrow$ elementary row operations (pivot element a_{kl})

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_k & \ldots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \ldots & \boldsymbol{a}_l & \ldots & \boldsymbol{a}_n & \boldsymbol{b}
\end{array} \\
\begin{bmatrix}
1 & 0 & \ldots & a'_{1k} & \ldots & 0 & a'_{1,m+1} & \ldots & 0 & \ldots & a'_{1n} & b'_1 \\
0 & 1 & \ldots & a'_{2k} & \ldots & 0 & a'_{2,m+1} & \ldots & 0 & \ldots & a'_{2n} & b'_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & a'_{kk} & \ldots & 0 & a'_{i,m+1} & \ldots & 1 & \ldots & a'_{kn} & b'_k \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & a'_{mk} & \ldots & 1 & a'_{m,m+1} & \ldots & 0 & \ldots & a'_{mn} & b'_m
\end{bmatrix}
\end{array}
$$

Elementary Row Operations

How to obtain another b.f.s of $\boldsymbol{Ax} = \boldsymbol{b}$ by changing the canonical form?

$$\begin{array}{c}
\begin{array}{ccccccccccccc}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_k & \dots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \dots & \boldsymbol{a}_l & \dots & \boldsymbol{a}_n & \boldsymbol{b}
\end{array} \\
\left[\begin{array}{ccccccccccccc}
1 & 0 & \dots & 0 & \dots & 0 & a_{1,m+1} & \dots & a_{1l} & \dots & a_{1n} & b_1 \\
0 & 1 & \dots & 0 & \dots & 0 & a_{2,m+1} & \dots & a_{2l} & \dots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & & 0 & a_{k,m+1} & \dots & \boxed{a_{kl}} & \dots & a_{kn} & b_k \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 0 & \dots & 1 & a_{m,m+1} & \dots & a_{ml} & \dots & a_{mn} & b_m
\end{array}\right]
\end{array}
\xrightarrow[\text{solution}]{\text{basic}}
\left[\begin{array}{c}
x_1 = b_1 \\
x_2 = b_2 \\
\vdots \\
x_m = b_m \\
x_{m+1} = 0 \\
\vdots \\
x_n = 0
\end{array}\right]$$

$\Downarrow$ elementary row operations (pivot element a_{kl})

$$\begin{array}{c}
\begin{array}{ccccccccccccc}
\boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_k & \dots & \boldsymbol{a}_m & \boldsymbol{a}_{m+1} & \dots & \boldsymbol{a}_l & \dots & \boldsymbol{a}_n & \boldsymbol{b}
\end{array} \\
\left[\begin{array}{ccccccccccccc}
1 & 0 & \dots & a'_{1k} & \dots & 0 & a'_{1,m+1} & \dots & 0 & \dots & a'_{1n} & b'_1 \\
0 & 1 & \dots & a'_{2k} & \dots & 0 & a'_{2,m+1} & \dots & 0 & \dots & a'_{2n} & b'_2 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & a'_{kk} & \dots & 0 & a'_{i,m+1} & \dots & 1 & \dots & a'_{kn} & b'_k \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & a'_{mk} & \dots & 1 & a'_{m,m+1} & \dots & 0 & \dots & a'_{mn} & b'_m
\end{array}\right]
\end{array}
\xrightarrow[\text{solution}]{\text{new basic}}
\left[\begin{array}{c}
x_1 = b'_1 \\
\vdots \\
x_k = 0 \\
\vdots \\
x_m = b'_m \\
x_{m+1} = 0 \\
x_l = b'_k \\
\vdots \\
x_n = 0
\end{array}\right]$$

Pivot Element

pivot equations:

when a_{kl} is fixed, the other elements are calculated via

- $b'_k = \frac{b_k}{a_{kl}}$
- $b'_i = b_i - \frac{b_k}{a_{kl}} a_{il}$ for $i = 1, 2, \cdots, m; i \neq k$.

pivot element

$\because$ b.f.s should satisfy $b_i > 0$, we expect $b'_i > 0$.

$\therefore$ a pivot element in the l-th column is opted by $\frac{b_k}{a_{kl}} = \min_{i=1,\cdots,m} \left\{ \frac{b_i}{a_{il}} \mid a_{il} > 0 \right\}$.

Pivot Element

Example

$$
\begin{aligned}
&\min\ -x_3 \\
&\text{s.t.}\quad x_1 \qquad\quad +3x_4 - x_5 + 3x_6 = 2 \\
&\qquad\qquad x_2 \quad +2x_4 - 2x_5 + x_6 = 1 \\
&\qquad\qquad\quad x_3 - 2x_4 - x_5 + 2x_6 = 3 \\
&\quad x_j \geq 0, j = 1, \cdots, 6
\end{aligned}
$$

Pivot Element

Example

$$\begin{aligned} \min\ & -x_3 \\ \text{s.t.}\ & x_1 \quad +3x_4 - x_5 + 3x_6 = 2 \\ & x_2 \quad +2x_4 - 2x_5 + x_6 = 1 \\ & x_3 - 2x_4 - x_5 + 2x_6 = 3 \\ & x_j \geq 0, j = 1, \cdots, 6 \end{aligned} \xrightarrow[\text{form}]{\text{standard}}$$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
1	0	0	3	-1	3	2
0	1	0	2	-2	1	1
0	0	1	-2	-1	2	3

$(x_1, x_2, x_3, x_4, x_5, x_6) = (2, 1, 3, 0, 0, 0)$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
1	0	0	3	-1	3	2
0	1	0	$\boxed{2}$	-2	1	1
0	0	1	-2	-1	2	3

$\Downarrow$elementary row operations with pivot

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
1	$-\frac{3}{2}$	0	0	2	$\frac{3}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	0	1	-1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	0	-3	3	4

$(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{1}{2}, 0, 4, \frac{1}{2}, 0, 0)$

idea of simplex method

find another solution from one solution until the optimal b.f.s is obtained.

Analysis: If $\boldsymbol{x}$ is a b.f.s, e.g.,

$$\boldsymbol{x} = [x_1, \cdots, x_m, 0, \cdots, 0]^\top, \ \ x_i \geq 0, \ i = 1, \cdots, m$$

$\therefore \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ implies that $x_1\boldsymbol{a}_1 + \cdots + x_m\boldsymbol{a}_m = \boldsymbol{b}$.

$$\text{standard form of LP:} \quad \min \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t.} \ \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}. \qquad (1)$$

Analysis: assume that the first m columns of $\boldsymbol{A}$ are the basis vectors, i.e.,
$\boldsymbol{A} = [\underbrace{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m}_{\boldsymbol{B} \in \mathbb{R}^{m \times m}}, \underbrace{\boldsymbol{a}_{m+1}, \cdots, \boldsymbol{a}_n}_{\boldsymbol{D} \in \mathbb{R}^{m \times (n-m)}}] = [\boldsymbol{B}, \boldsymbol{D}]$.

$\therefore \boldsymbol{x} = [\underbrace{x_1, \cdots, x_m}_{\boldsymbol{x_B} \in \mathbb{R}^m}, \underbrace{x_{m+1}, \cdots, x_n}_{\boldsymbol{x_D} \in \mathbb{R}^{n-m}}] = [\boldsymbol{x_B}, \boldsymbol{x_D}]$ and $\boldsymbol{c}^\top = [\boldsymbol{c}_{\boldsymbol{B}}^\top, \boldsymbol{c}_{\boldsymbol{D}}^\top]$.

Matrix Form of the Simplex Method

(1) $\Longleftrightarrow$ $\boxed{\begin{array}{ll} \min & \boldsymbol{c}_B^\top \boldsymbol{x}_B + \boldsymbol{c}_D^\top \boldsymbol{x}_D \\ \text{s.t.} & \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{D}\boldsymbol{x}_D = \boldsymbol{b},\ \boldsymbol{x}_B \geq \boldsymbol{0}, \boldsymbol{x}_D \geq \boldsymbol{0}. \end{array}}$

- If $\boldsymbol{x}_D = 0$, then $\boldsymbol{x} = [\boldsymbol{x}_B, \boldsymbol{x}_D]$ is a b.f.s,
  $\therefore \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_D \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \xrightarrow[\text{fun. val.}]{\text{objective}} \boldsymbol{z}_0 = \boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b}.$
- If $\boldsymbol{x}_D \neq 0$, then $\boldsymbol{x} = [\boldsymbol{x}_B, \boldsymbol{x}_D]$ is not a b.f.s,
  $\therefore \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D.$
  Thus, the objective function value is

$$\begin{aligned} \boldsymbol{z} &= \boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{c}_B^\top \boldsymbol{x}_B + \boldsymbol{c}_D^\top \boldsymbol{x}_B \\ &= \boldsymbol{c}_B^\top \left(\boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{D}\boldsymbol{x}_D\right) + \boldsymbol{c}_D^\top \boldsymbol{x}_D \\ &= \underbrace{\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b}}_{\boldsymbol{z}_0} + \underbrace{\left(\boldsymbol{c}_D^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D}\right)}_{=:\boldsymbol{r}_D^\top} \boldsymbol{x}_D = \boldsymbol{z}_0 + \boldsymbol{r}_D^\top \boldsymbol{x}_D \qquad (2) \\ &= \boldsymbol{z}_0 + \sigma_{m+1}x_{m+1} + \cdots + \sigma_i x_i + \cdots + \sigma_n x_n \end{aligned}$$

where $\boldsymbol{r}_D = [\sigma_{m+1}, \cdots, \sigma_n]^\top$ and $\boldsymbol{x}_D = [x_{m+1}, \cdots, x_n]^\top$.

Discussion on Optimal Solution

1. If $\boldsymbol{r_D} \geq 0$, then $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ is a b.s.

   Moreover, if $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \geq 0$, then $\boldsymbol{x}$ is an optimal b.f.s. The optimal value is $\boldsymbol{z}_0 = \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b}$.

2. If $\boldsymbol{r_D}$ has negative entry, e.g., $\sigma_i < 0$, then $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ is not optimal.

$$\therefore (2) \Leftrightarrow \boldsymbol{z} = \boldsymbol{z}_0 + \sigma_{m+1}x_{m+1} + \cdots + \sigma_i x_i + \cdots + \sigma_n x_n. \tag{3}$$

   Obviously, the objective function value can be reduced if we choose the i-th column $\boldsymbol{a}_i$ as basis. It means row operations with a pivot element at i-th column can render smaller function value.

Example (whether $\boldsymbol{x} = (4, 0, 5, 0)$ is a minimizer of LP?)

$$\begin{array}{rl} \min & x_1 + 2x_2 + 3x_3 - 2x_4 \\ \text{s.t.} & x_1 - 2x_2 + x_4 = 4 \\ & x_2 + x_3 - 2x_4 = 5 \\ & x_i \geq 0, i = 1, \cdots, 4. \end{array} \Longrightarrow \text{tips: check the discriminant } \boldsymbol{r_D} \geq 0$$

Matrix Form of the Simplex Method

Matrix Form of the Simplex Method

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c}^\top & 0 \end{bmatrix} \xlongequal{\text{blocky case}} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{b} \\ \boldsymbol{c}_B^\top & \boldsymbol{c}_D^\top & 0 \end{bmatrix}$$

$$\xRightarrow{\text{1st-row premultiply } \boldsymbol{B}^{-1}} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{c}_B^\top & \boldsymbol{c}_D^\top & 0 \end{bmatrix}$$

$$\xRightarrow{-\boldsymbol{c}_B^\top \times \text{1st-row} + \text{2nd row}} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{B}^{-1}\boldsymbol{D} & \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0}^\top & \boldsymbol{c}_D^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{D} & -\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{b} \end{bmatrix}$$

★ basis, basic variable, basic feasible solution, discriminant and objective function value are all in the last matrix.

Matrix Form of the Simplex Method

operation of simplex table

- if $\boldsymbol{r}_{\boldsymbol{D}} = \boldsymbol{c}_{\boldsymbol{D}}^{\top} - \boldsymbol{c}_{\boldsymbol{B}}^{\top}\boldsymbol{B}^{-1}\boldsymbol{D} \geq 0$, then $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \mathbf{0} \end{bmatrix}$ is b.s, and also the minimizer. Furthermore, $\boldsymbol{z}^* = \boldsymbol{c}_{\boldsymbol{B}}^{\top}\boldsymbol{B}^{-1}\boldsymbol{b}$ is optimal value.
- otherwise, choose the column has the smallest element of $\boldsymbol{r}_{\boldsymbol{D}}$ and select a pivot to carry out elementary row operation.

Simplex Method

Example (solve LP by simplex method)

$$
\begin{aligned}
\max \quad & 7x_1 + 6x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \le 3 \\
& x_1 + 4x_2 \le 4 \\
& x_1 \ge 0, x_2 \ge 0
\end{aligned}
$$

Simplex Method

Example (solve LP by simplex method)

$$\begin{aligned} \max\ & 7x_1 + 6x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \le 3 \\ & x_1 + 4x_2 \le 4 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned} \quad \xrightarrow[\text{form}]{\text{standard}} \quad \begin{aligned} \min\ & -7x_1 - 6x_2 \\ \text{s.t.}\ & 2x_1 + x_2 + x_3 = 3 \\ & x_1 + 4x_2 + x_4 = 5 \\ & x_i \ge 0, i = 1, \dots, 4 \end{aligned}$$

step 1: initial simplex

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
$\boxed{2}$	1	1	0	3
1	4	0	1	4
-7	-6	0	0	0

Basis: $\boldsymbol{a}_1, \boldsymbol{a}_2$; Basic variable: x_3, x_4,

Basic feasible solution:

$(x_1, x_2, x_3, x_4) = (0, 0, 3, 4)$,

Discriminant: $\boldsymbol{r_D} = (-7, -6)$,

Objective function: 0.

step 2: pivot operation on $\boxed{2}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
0	$\boxed{\frac{7}{2}}$	$-\frac{1}{2}$	1	$\frac{5}{2}$
0	$-\frac{5}{2}$	$\frac{7}{2}$	0	$\frac{21}{2}$

Basis: $\boldsymbol{a}_1, \boldsymbol{a}_4$; Basic variable: x_1, x_4,

Basic feasible solution:

$(x_1, x_2, x_3, x_4) = (\frac{3}{2}, 0, 0, \frac{5}{2})$,

Discriminant: $\boldsymbol{r_D} = (-\frac{5}{2}, \frac{7}{2})$,

Objective function: $-\frac{21}{2}$.

step 3: pivot operation on $\boxed{\frac{7}{2}}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{b}$
1	0	$\frac{4}{7}$	$-\frac{1}{7}$	$\frac{8}{7}$
0	1	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{5}{7}$
0	0	$\frac{22}{7}$	$\frac{5}{7}$	$\frac{86}{7}$

Basis: $\boldsymbol{a}_1, \boldsymbol{a}_2$; Basic variable: x_1, x_2,

Basic feasible solution:
$(x_1, x_2, x_3, x_4) = (\frac{8}{7}, \frac{5}{7}, 0, 0)$,

Discriminant: $\boldsymbol{r_D} = (\frac{22}{7}, \frac{5}{7}) > 0$,
which implies $\boldsymbol{x}$ is an optimal solution.

Objective function: $-\frac{86}{7}$.

Example (solve LP by simplex method)

$$
\begin{array}{llllllll}
\min & x_2 & -3x_3 & +2x_5 & & & & \\
\text{s.t.} & x_1 & +3x_2 & -x_3 & & +2x_5 & & = 3, \\
& & -2x_2 & +4x_3 & +x_4 & & & = 5, \\
& & -4x_2 & +3x_3 & & +8x_5 & +x_6 & = 5, \quad x_i \geq 0, i = 1, 2, \cdots, 6.
\end{array}
$$

step 1: initial simplex

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
1	3	−1	0	2	0	7
0	−2	$\boxed{4}$	1	0	0	12
0	−4	3	0	8	1	10
0	1	−3	0	2	0	0

Basis: $\boldsymbol{a}_1, \boldsymbol{a}_4, \boldsymbol{a}_6$; Basic variable: x_1, x_4, x_6,

Basic feasible solution: $(x_1, x_2, x_3, x_4, x_5, x_6) = (7, 0, 0, 12, 0, 10)$,

Discriminant: $\boldsymbol{r_D} = (1, -3, 2)$,

Objective function: 0.

step 2: pivot operation on $\boxed{4}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
1	$\boxed{\frac{5}{2}}$	0	$\frac{1}{4}$	2	0	10
0	-1	1	$\frac{1}{4}$	0	0	3
0	$-\frac{5}{2}$	0	$-\frac{3}{4}$	8	1	1
0	$-\frac{1}{2}$	0	$\frac{3}{4}$	2	0	9

Basis: $\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_6$,

Basic variable: x_1, x_3, x_6,

Basic feasible solution: $(x_1, x_2, x_3, x_4, x_5, x_6) = (10, 0, 3, 0, 0, 1)$,

Discriminant: $\boldsymbol{r_D} = (-\frac{1}{2}, \frac{3}{4}, 2)$,

Objective function: -9.

step 3: pivot operation on $\boxed{\frac{5}{2}}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
$\frac{2}{5}$	1	0	$\frac{1}{10}$	$\frac{4}{5}$	0	4
$\frac{1}{5}$	0	1	$\frac{3}{10}$	$\frac{2}{5}$	0	5
1	0	0	$-\frac{1}{2}$	10	1	11
$\frac{1}{5}$	0	0	$\frac{4}{5}$	$\frac{12}{5}$	0	11

Basis: $\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_6$,

Basic variable: x_2, x_3, x_6,

Basic feasible solution: $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 4, 5, 0, 0, 11)$,

Discriminant: $\boldsymbol{r_D} = (\frac{1}{5}, \frac{4}{5}, \frac{12}{5}) > 0$, which implies $\boldsymbol{x}$ is an optimal solution.

Objective function: -11.

Initial B.F.S.

- ★ Simplex method starts with a tableau for standard form of LP, namely, it requires an initial b.f.s.
- ★ How to find an initial b.f.s.? Choose any m basis columns? (ans: ✗)

Example (some LP has obvious initial b.f.s)

The LP with strict inequality $\boldsymbol{Ax} \le \boldsymbol{b}$ can be formalized as standard form by introducing m slack variables $z_i > 0$,

$$\begin{aligned} &\boldsymbol{Ax} \le \boldsymbol{b} \\ &\boldsymbol{x} \ge 0 \end{aligned}$$

Initial B.F.S.

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The LP with strict inequality $\boldsymbol{Ax} \le \boldsymbol{b}$ can be formalized as standard form by introducing m slack variables $z_i > 0$,

$$\begin{array}{l} \boldsymbol{Ax} \le \boldsymbol{b} \\ \boldsymbol{x} \ge 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \boldsymbol{Ax} + \boldsymbol{z} = \boldsymbol{b} \\ \boldsymbol{x} \ge 0, \ \boldsymbol{z} \ge 0 \end{array}$$

Initial B.F.S.

- ★ Simplex method starts with a tableau for standard form of LP, namely, it requires an initial b.f.s.
- ★ How to find an initial b.f.s.? Choose any m basis columns? (ans: ✗)

Example (some LP has obvious initial b.f.s)

The LP with strict inequality $\boldsymbol{Ax} \leq \boldsymbol{b}$ can be formalized as standard form by introducing m slack variables $z_i > 0$,

$$\begin{array}{l} \boldsymbol{Ax} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq 0 \end{array} \Longrightarrow \begin{array}{l} \boldsymbol{Ax} + \boldsymbol{z} = \boldsymbol{b} \\ \boldsymbol{x} \geq 0, \ \boldsymbol{z} \geq 0 \end{array} \Longleftrightarrow [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} = \boldsymbol{b}, \ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix} \geq \boldsymbol{0}, \tag{4}$$

where $\boldsymbol{z} = [z_1, \cdots, z_m]^\top$ is slack variable.

Initial B.F.S.

- ★ Simplex method starts with a tableau for standard form of LP, namely, it requires an initial b.f.s.
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Example (some LP has obvious initial b.f.s)

The LP with strict inequality $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ can be formalized as standard form by introducing m slack variables $z_i > 0$,

$$\begin{array}{l}\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ \boldsymbol{x} \ge 0\end{array} \implies \begin{array}{l}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{z} = \boldsymbol{b} \\ \boldsymbol{x} \ge 0, \ \boldsymbol{z} \ge 0\end{array} \iff [\boldsymbol{A}, \boldsymbol{I}_m]\begin{bmatrix}\boldsymbol{x} \\ \boldsymbol{z}\end{bmatrix} = \boldsymbol{b}, \ \begin{bmatrix}\boldsymbol{x} \\ \boldsymbol{z}\end{bmatrix} \ge \boldsymbol{0}, \qquad (4)$$

where $\boldsymbol{z} = [z_1, \cdots, z_m]^\top$ is slack variable.

Obviously, right-hand of (4) has a b.f.s $\begin{bmatrix}\boldsymbol{x} \\ \boldsymbol{z}\end{bmatrix} = \begin{bmatrix}0 \\ \boldsymbol{b}\end{bmatrix}$.

Initial B.F.S.

original LP: (P0)

$$
\begin{aligned}
\min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

Initial B.F.S.

original LP: (P0)

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \quad \xrightarrow[\boldsymbol{e}=(1,1,\cdots,1)^\top \in \mathbb{R}^m]{\text{artificial variable } \boldsymbol{y} \in \mathbb{R}^m}$$

original LP: (P0)

$$\min \ \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t.} \ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}$$

$$\xrightarrow[\boldsymbol{e}=(1,1,\cdots,1)^\top \in \mathbb{R}^m]{\text{artificial variable } \boldsymbol{y} \in \mathbb{R}^m}$$

artificial LP: (P1)

$$\min \ y_1 + y_2 + \cdots + y_m = \boldsymbol{e}^\top \boldsymbol{y}$$

$$\text{s.t.} \ [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{y} \geq \boldsymbol{0}.$$

Proposition

$(\mathrm{P0})$ *has a b.f.s* $\Longleftrightarrow$ $(\mathrm{P1})$ *has a minimizer with optimal value as 0.*

proof. $\Rightarrow$) if (P0) has a b.f.s $\boldsymbol{x}$, then $[\boldsymbol{x}^\top, \boldsymbol{0}^\top]^\top$ is a b.f.s of (P1) with function value as 0 (why?)

$\therefore$ $[\boldsymbol{x}^\top, \boldsymbol{0}^\top]^\top$ is a minimizer of (P1).

$\Leftarrow$) if (P1) has an optimal f.s. with objective function value 0, then this solution must have the form $[\boldsymbol{x}^\top, \boldsymbol{0}^\top]^\top$ (why?), where $\boldsymbol{x} \geq \boldsymbol{0}$

$\therefore$ $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and $\boldsymbol{x}$ is a f.s. of (P0).

Two-Phase Simplex Method

idea of two-phase simplex method

assume (P0) has some f.s, then find a minimizer of (P1).

- find the optimal feasible solution of (P1), denoted by $\begin{bmatrix} \boldsymbol{x} \\ \mathbf{0} \end{bmatrix}$.
- if $\begin{bmatrix} \boldsymbol{x} \\ \mathbf{0} \end{bmatrix}$ is nondegeneracy and the basic variables are in the first n components (none of the artificial variables are basic), then $\boldsymbol{x}$ is a b.f.s of (P0). Use this b.f.s as the initial b.f.s of (P0).

Two-Phase Simplex Method

Phase I: construct (P1), then solve (P1) by simplex method

(P0)

$$\begin{aligned} \min\ & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Two-Phase Simplex Method

Phase I: construct (P1), then solve (P1) by simplex method

(P0)

$$\begin{aligned} \min\ & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \Longrightarrow$$

(P1)

$$\begin{aligned} \min\ & y_1 + y_2 + \cdots + y_m = \boldsymbol{e}^\top \boldsymbol{y} \\ \text{s.t. } & [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned}$$

Two-Phase Simplex Method

Phase I: construct (P1), then solve (P1) by simplex method

(P0)

$$\begin{aligned} \min\ & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \Longrightarrow$$

(P1)

$$\begin{aligned} \min\ & y_1 + y_2 + \cdots + y_m = \boldsymbol{e}^\top \boldsymbol{y} \\ \text{s.t. } & [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned} \Longrightarrow$$

initial simplex of (P1)

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{b}$
$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$\boldsymbol{0}$	$\boldsymbol{e}$	0

Two-Phase Simplex Method

Phase I: construct (P1), then solve (P1) by simplex method

(P0)

$$\min \boldsymbol{c}^\top \boldsymbol{x} \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$

$\Longrightarrow$ (P1)

$$\min\ y_1 + y_2 + \cdots + y_m = \boldsymbol{e}^\top \boldsymbol{y}$$

$$\text{s.t. } [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}.$$

$\Longrightarrow$ initial simplex of (P1) $\Longrightarrow$

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{b}$
$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$\boldsymbol{0}$	$\boldsymbol{e}$	$\boldsymbol{0}$

final simplex of (P1)

x_1	x_2	$\cdots$	x_k	$\cdots$	x_n	y_1	$\cdots$	y_l	$\cdots$	y_m	$\boldsymbol{b}$
a'_{11}	a'_{12}	$\cdots$	a'_{1k}	$\cdots$	a'_{1n}	d'_{11}	$\cdots$	d'_{1l}	$\cdots$	d'_{1m}	b'_1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{i1}	a'_{i2}	$\cdots$	a'_{ik}	$\cdots$	a'_{in}	d'_{i1}	$\cdots$	d'_{il}	$\cdots$	d'_{im}	b'_i
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{m1}	a'_{m2}	$\cdots$	a'_{mk}	$\cdots$	a'_{mn}	d'_{m1}	$\cdots$	d'_{ml}	$\cdots$	d'_{mm}	b'_m
r'_1	r'_2	$\cdots$	r'_k	$\cdots$	r'_n	s'_1	$\cdots$	s'_l	$\cdots$	s'_m	$-f^*$

$(*)$

Two-Phase Simplex Method

Phase I: construct (P1), then solve (P1) by simplex method

(P0)

$$\begin{aligned} &\min\ \boldsymbol{c}^\top \boldsymbol{x} \\ &\text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ &\quad \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \Longrightarrow$$

(P1)

$$\begin{aligned} &\min\ y_1 + y_2 + \cdots + y_m = \boldsymbol{e}^\top \boldsymbol{y} \\ &\text{s.t. } [\boldsymbol{A}, \boldsymbol{I}_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{y} \geq \boldsymbol{0}. \end{aligned} \Longrightarrow$$

initial simplex of (P1)

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{b}$
$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
$\boldsymbol{0}$	$\boldsymbol{e}$	$\boldsymbol{0}$

$\Longrightarrow$

final simplex of (P1)

x_1	x_2	$\cdots$	x_k	$\cdots$	x_n	y_1	$\cdots$	y_l	$\cdots$	y_m	$\boldsymbol{b}$
a'_{11}	a'_{12}	$\cdots$	a'_{1k}	$\cdots$	a'_{1n}	d'_{11}	$\cdots$	d'_{1l}	$\cdots$	d'_{1m}	b'_1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{i1}	a'_{i2}	$\cdots$	a'_{ik}	$\cdots$	a'_{in}	d'_{i1}	$\cdots$	d'_{il}	$\cdots$	d'_{im}	b'_i
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{m1}	a'_{m2}	$\cdots$	a'_{mk}	$\cdots$	a'_{mn}	d'_{m1}	$\cdots$	d'_{ml}	$\cdots$	d'_{mm}	b'_m
r'_1	r'_2	$\cdots$	r'_k	$\cdots$	r'_n	s'_1	$\cdots$	s'_l	$\cdots$	s'_m	$-f^*$

$(*)$

- If $f^* > 0$, then (P0) has no f.s, i.e., constraint of (P0) is empty.
- If $f^* = 0$, proceed to following Phase II.

Two-Phase Simplex Method

Phase II:

- Case 1: if the basic variables in table (*) are all in the component of $\boldsymbol{x}$, then delete all $\boldsymbol{y}$ in table (*), and replace the discriminant with the objective function vector $\boldsymbol{c}$ of (P0), and then use simplex method again.

x_1	x_2	$\cdots$	x_k	$\cdots$	x_n	$\boldsymbol{b}$
a'_{11}	a'_{12}	$\cdots$	a'_{1k}	$\cdots$	a'_{1n}	b'_1
a'_{21}	a'_{22}	$\cdots$	a'_{2k}	$\cdots$	a'_{2n}	b'_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{i1}	a'_{i2}	$\cdots$	a'_{ik}	$\cdots$	a'_{in}	b'_i
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{m1}	a'_{m2}	$\cdots$	a'_{mk}	$\cdots$	a'_{mn}	b'_m
c'_1	c'_2	$\cdots$	c'_k	$\cdots$	c'_n	0

Two-Phase Simplex Method

Phase II:

- Case 2: some basic variables in table (*) are in the component of $\boldsymbol{y}$, e.g., y_l.
  - Case 2.1: if $a_{ij} = 0$ for all $j = 1, 2, \cdots, n$, then delete the i-th row.
  - Case 2.2: otherwise, set any $a_{ij} \neq 0$ as pivot element.

x_1	x_2	$\cdots$	x_k	$\cdots$	x_n	y_1	$\cdots$	y_l	$\cdots$	y_m	$\boldsymbol{b}$
a'_{11}	a'_{12}	$\cdots$	a'_{1k}	$\cdots$	a'_{1n}	d'_{11}	$\cdots$	0	$\cdots$	d'_{1m}	b'_1
a'_{21}	a'_{22}	$\cdots$	a'_{2k}	$\cdots$	a'_{2n}	d'_{21}	$\cdots$	0	$\cdots$	d'_{2m}	b'_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{i1}	a'_{i2}	$\cdots$	a'_{ik}	$\cdots$	a'_{in}	d'_{i1}	$\cdots$	1	$\cdots$	d'_{im}	b'_l
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
a'_{m1}	a'_{m2}	$\cdots$	a'_{mk}	$\cdots$	a'_{mn}	d'_{m1}	$\cdots$	0	$\cdots$	d'_{mm}	b'_m
r'_1	r'_2	$\cdots$	r'_k	$\cdots$	r'_n	s'_1	$\cdots$	0	$\cdots$	s'_m	$-f^*$