15 Introduction to Linear Programming

Linear Program

Example (what is the plan of carpenter?)

A carpenter tries to maximize the revenue of his factory. He can make: (A) bookshelf, (B) door, (C) desk and (D) chair. The following Table gives the quantities of materials, labor and revenue earned for each type of furniture. Suppose that 5000 units of wood and 1500 units of labor are available, how many furniture he should make?

furniture	wood	labor	revenue	furniture	wood	labor	revenue
(A) bookshelf	10	2	100	(B) door	12	4	150
(C) desk	25	8	200	(D) chair	20	12	400

Ans: assume the number of bookshelf$= x_1$, door$= x_2$, desk$= x_3$, and chair$= x_4$.

$$
\begin{array}{rr}
\text{maximize revenue} \Rightarrow & \max 100x_1 + 150x_2 + 200x_3 + 400x_4 \\
\text{under conditions} \Rightarrow & 10x_1 + 12x_2 + 25x_3 + 20x_4 \leq 5000 \\
& 2x_1 + 4x_2 + 8x_3 + 12x_4 \leq 1500 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.
\end{array}
\Rightarrow
\begin{array}{ll}
\max & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\
& \boldsymbol{x} \geq 0
\end{array}
$$

Goal: linear program is to minimize/maximize a linear objective function $f(\boldsymbol{x}) = \boldsymbol{c}^\top \boldsymbol{x}$ subject to constraints represented by linear equality and/or inequality.

Definition (linear program, LP)

$$\boxed{\min \boldsymbol{c}^\top \boldsymbol{x}, \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0},$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$.

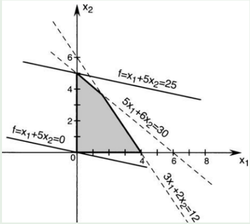
- ★ variations of LP: minimize $\Rightarrow$ maximize; $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ or $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$; $\boldsymbol{x} \geq 0 \Rightarrow \boldsymbol{x} \in \mathbb{R}^n$; $\cdots$
- ★ if $\boldsymbol{x} \geq 0$ is replaced by $\boldsymbol{x} \geq 0, \boldsymbol{x} \in \mathbb{Z}^n$, it is a integral program (Chapt 19).
- ★ many examples of LP from applications (see textbook).

Linear Program

Example (two-dimensional LP)

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \\ & \boldsymbol{x} \ge 0 \end{aligned}$$

- $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$
- $\boldsymbol{A} = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 30 \\ 20 \end{bmatrix}$



- dash lines: inequality constraint
- real line: parallel contours of objective function.
- dark region: constraint
- analysis the maximizer by contours of objective.

maximizer=$\boldsymbol{x}^* = [0, 5]^\top$

maximal value $f^* = \boldsymbol{c}^\top \boldsymbol{x}^* = 25$

exercise: solve two-dimensional LP geometrically

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \ge \boldsymbol{b}, \text{ where } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}. \\ & \boldsymbol{x} \ge 0, \end{aligned}$$

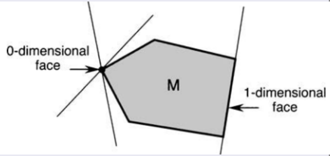
Ans: maximizer=$\boldsymbol{x}^* = [0, 50]^\top$ and maximal value $f^* = \boldsymbol{c}^\top \boldsymbol{x}^* = 50$.

Linear Program

constraint: e.g., $M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\}$

M is the intersection of a finite number of closed halfspaces $\Longrightarrow$ M is a convex polytope.
Let H be a hyperplane supporting M.

- if $\dim(M) \leq n \Rightarrow H \cap M = M$.
- if $\dim(M) = n \Rightarrow H \cap M =: F_{H,M} \subset M$ is face.
- if $\dim(F_{H,M}) = (n-1) \Rightarrow \exists \mid H$ supporting M.

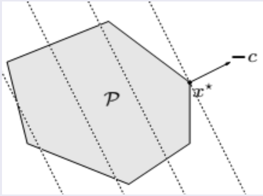


- if $\dim(F_{H,M}) \leq (n-1) \Rightarrow \exists$ an infinite number of H supporting M.

properties of LP

LP $\Longleftrightarrow$ minimize/maximize linear function on polyhedron.

- if $M = \emptyset$, then LP has no optimum.
- if M is compact (closed+bounded), then LP has optima (possible not unique).
- if M is unbounded, uncertain to has optima.



Definition (standard form of LP)

$\min\ \boldsymbol{c}^\top \boldsymbol{x}$, s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$.
Require: $\mathrm{rank}(\boldsymbol{A}) = m$ and $\boldsymbol{b} \geq 0$. (why?)

★ how to formulate LP as standard form?

- $\max \boldsymbol{c}^\top \boldsymbol{x} \Longrightarrow \min(-\boldsymbol{c})^\top \boldsymbol{x}$
- introduce nonnegative variable to formulate inequality as equality, i.e., $\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i \Longrightarrow \boldsymbol{a}_i^\top \boldsymbol{x} + y_i = b_i$ or $\boldsymbol{a}_i^\top \boldsymbol{x} \geq b_i \Longrightarrow \boldsymbol{a}_i^\top \boldsymbol{x} - y_i = b_i$.
- $x_i \in \mathbb{R} \Longrightarrow x_i = u + v, u \geq 0, v \geq 0.$
- $b_i < 0$, multiply ith equality by -1.

Linear Program

Examples

- inequality $x_1 \le 7 \xrightarrow{\text{introduce } x_2} \begin{cases} x_1 + x_2 = 7, \\ x_2 \ge 0. \end{cases}$
- let a_1, a_2, b be positive scalars. The inequality
$$\begin{cases} a_1x_1 + a_2x_2 \le b, \\ x_1 \ge 0, \quad x_2 \ge 0. \end{cases} \xrightarrow{\text{introduce } x_3} \begin{cases} a_1x_1 + a_2x_2 + x_3 = b, \\ x_1 \ge 0, \ x_2 \ge 0, \ x_3 \ge 0. \end{cases}$$
- $\begin{cases} \boldsymbol{Ax} \ge \boldsymbol{b}, \\ \boldsymbol{x} \ge 0 \end{cases} \xrightarrow{\text{introduce } \boldsymbol{y} \in \mathbb{R}^m} \begin{cases} \boldsymbol{Ax} - \boldsymbol{y} = \boldsymbol{b}, \\ \boldsymbol{x} \ge 0, \ \boldsymbol{y} \ge 0. \end{cases}$
- $\begin{cases} \max & -x_1 + x_2 \\ \text{s.t.} & 3x_1 - x_2 = -5, \\ & |x_2| \le 2, \\ & x_1 \le 0. \end{cases} \xrightarrow{\text{why?}} \begin{cases} \min & -x_1' - u + v \\ \text{s.t.} & 3x_1' + u - v = 5, \\ & u + v + x_3 = 2, \\ & x_1' \ge 0, \ u \ge 0, \ v \ge 0, \ x_3 \ge 0. \end{cases}$

Standard Form of LP

standard form of LP

$\min\ \boldsymbol{c}^\top \boldsymbol{x}$, s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq 0$,
where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\text{rank}(\boldsymbol{A}) = m$ and $\boldsymbol{b} \geq 0$.

Analysis of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$:

let $\boldsymbol{A} = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m, \boldsymbol{a}_{m+1}, \cdots, \boldsymbol{a}_n]$.

$\because$ $\text{rank}(\boldsymbol{A}) = m$,

by choosing m linearly independent columns (w.l.o.g) $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$,

$$\therefore\ \boldsymbol{A} = [\underbrace{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m}_{\boldsymbol{B}\in\mathbb{R}^{m\times m}}, \underbrace{\boldsymbol{a}_{m+1}, \cdots, \boldsymbol{a}_n}_{\boldsymbol{D}\in\mathbb{R}^{m\times(n-m)}}] = [\boldsymbol{B}, \boldsymbol{D}].$$

$$\text{Let } \boldsymbol{x} = [\underbrace{x_1, \cdots, x_m}_{\boldsymbol{x_B}\in\mathbb{R}^{m}}, \underbrace{x_{m+1}, \cdots, x_n}_{\boldsymbol{x_D}\in\mathbb{R}^{(n-m)}}] = [\boldsymbol{x_B}, \boldsymbol{x_D}].$$

$$\therefore\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{a}_1 x_1 + \boldsymbol{a}_2 x_2 + \cdots + \boldsymbol{a}_n x_n = b \iff \boldsymbol{B}\boldsymbol{x_B} + \boldsymbol{D}\boldsymbol{x_D} = \boldsymbol{b}.$$

$\therefore$ $\boldsymbol{x} = [\boldsymbol{x_B}, \boldsymbol{x_D}] = [\boldsymbol{B}^{-1}\boldsymbol{b}, 0]$ is a solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Standard Form of LP

Definition (basic solution)

$[\boldsymbol{x_B}, \boldsymbol{x_D}] = [\boldsymbol{B}^{-1}\boldsymbol{b}, 0]$ is called a basic solution to $\boldsymbol{Ax} = \boldsymbol{b}$ w.r.t to basis $\boldsymbol{B}$.

$\boldsymbol{x_B}$: basic variables; $\boldsymbol{B}$: basic matrix/columns;

$\boldsymbol{x_D}$: nonbasic variables; $\boldsymbol{D}$: nonbasic matrix/columns.

★ if $\boldsymbol{B}^{-1}\boldsymbol{b}$ has 0 entry, $[\boldsymbol{x_B}, \boldsymbol{x_D}] = [\boldsymbol{B}^{-1}\boldsymbol{b}, 0]$ is called degenerate basic solution.

Standard Form of LP

Recall the constraint of stardard LP: $M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\}$.

Definition (feasible solution to M)

feasible solution to M: $\forall \boldsymbol{x} \in M$.
basic feasible solution to M: basic solution in M, i.e., $[\boldsymbol{x_B}, \boldsymbol{x_D}] = [\boldsymbol{B}^{-1}\boldsymbol{b}, 0] \geq 0$.

★ how to get basic feasible solution to M?

$$\xRightarrow[\text{-dent columns of } \boldsymbol{A}]{\text{linear indepen}} \boldsymbol{B} = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m]$$

$$\xRightarrow[\text{solution}]{\text{basic}} \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ 0 \end{bmatrix} \xRightarrow{\text{nonnegative?}} \text{Yes}$$

Standard Form of LP

Example ($M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0\}$)

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

whether feasible solution (f.s)? basic solution (b.s)? basic feasible solution (b.f.s)?

1. $\boldsymbol{x} = [6, 2, 0, 0]$, $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2] \Rightarrow \boldsymbol{B}^{-1}\boldsymbol{b} = [6, 2] \Rightarrow$ f.s+b.s $\Rightarrow$ b.f.s
2. $\boldsymbol{x} = [0, 0, 0, 2]$, $\boldsymbol{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4] \Rightarrow \boldsymbol{B}^{-1}\boldsymbol{b} = [0, 2] \Rightarrow$f.s+b.s $\Rightarrow$ b.f.s(degenerate)
3. $\boldsymbol{x} = [3, 1, 0, 1] \Rightarrow$ nonzeros$= 3 \neq 2 \Longrightarrow$ f.s, not b.s
4. $\boldsymbol{x} = [0, 2, -6, 0]$,$\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3] \Rightarrow \boldsymbol{B}^{-1}\boldsymbol{b} = [2, -6] \Rightarrow$b.s, not b.f.s

Standard Form of LP

Example ($M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0\}$)

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 4 & 1 & 1 & -2 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$$

choose any linear independent column with number of rank($\boldsymbol{A}$).

1. $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2] \Rightarrow \begin{cases} \boldsymbol{x}_B = [x_1, x_2] = \boldsymbol{B}^{-1}\boldsymbol{b} = [\frac{14}{5}, -\frac{11}{5}] \\ \boldsymbol{x}_D = [x_3, x_4] = [0, 0] \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = [\frac{14}{5}, -\frac{11}{5}, 0, 0] \\ f.s(✗) \end{cases}$
2. $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_3] \Rightarrow \begin{cases} \boldsymbol{x}_B = [x_1, x_3] = \boldsymbol{B}^{-1}\boldsymbol{b} = [\frac{4}{3}, \frac{11}{3}] \\ \boldsymbol{x}_D = [x_3, x_4] = [0, 0] \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = [\frac{4}{3}, 0, \frac{11}{3}, 0] \\ f.s(✓) \end{cases}$
3. $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_4]$, linear independent (✗), not a basis.
4. $\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3] \Rightarrow \begin{cases} \boldsymbol{x}_B = [x_2, x_3] = \boldsymbol{B}^{-1}\boldsymbol{b} = [2, 7] \\ \boldsymbol{x}_D = [x_1, x_4] = [0, 0] \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = [0, 2, 7, 0] \\ f.s(✓) \end{cases}$
5. $\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_4] \Rightarrow \begin{cases} \boldsymbol{x}_B = [x_2, x_4] = \boldsymbol{B}^{-1}\boldsymbol{b} = [\frac{-11}{5}, \frac{-28}{5}] \\ \boldsymbol{x}_D = [x_3, x_4] = [0, 0] \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = [0, \frac{-11}{5}, 0, \frac{-28}{5}] \\ f.s(✗) \end{cases}$
6. $\boldsymbol{B} = [\boldsymbol{a}_3, \boldsymbol{a}_4] \Rightarrow \begin{cases} \boldsymbol{x}_B = [x_3, x_4] = \boldsymbol{B}^{-1}\boldsymbol{b} = [\frac{11}{3}, \frac{-8}{3}] \\ \boldsymbol{x}_D = [x_1, x_2] = [0, 0] \end{cases} \Rightarrow \begin{cases} \boldsymbol{x} = [0, 0, \frac{11}{3}, \frac{-8}{3}] \\ f.s(✗) \end{cases}$

Lemma ($M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0\}$)

If $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, then the number of basic solution is less than $C_n^m = \frac{n!}{m!(n-m)!}$.

★ Unfortunately, e.g., $C_{50}^5 = 2118760!$

Definition (optimal feasible solution)

$\boldsymbol{x}$ is an optimal feasible solution if it yields the minimum value of the objective function $\boldsymbol{c}^\top \boldsymbol{x}$ over the constraints M.

Theorem (fundamental theorem of LP in standard form)

- *if there exists f.s, then there exists b.f.s.*
- *if there exists an optimal f.s, then there exists an optimal b.f.s.*

proof. let $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]^\top$ an f.s with p positive entries.
(w.l.o.g) assume $\{x_i > 0\}_{i=1}^p$, and $\{x_i = 0\}_{i=p+1}^n$.

$$\therefore\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \iff \boldsymbol{a}_1 x_1 + \boldsymbol{a}_2 x_2 + \cdots + \boldsymbol{a}_p x_p = b. \quad (1)$$

Solution of LP

- case 1: If $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p$ are linearly independent, then $p < m$.
  If $p = m$, then $\boldsymbol{x}$ is b.f.s. [proof is done].
  If $p < m$, $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p$ can be enlarged to m columns as a basis of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$. $\Longrightarrow \boldsymbol{x}$ is a degenerate b.f.s w.r.t. the enlarged basis.
- case 2: If $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p$ are linearly dependent, then

$$\exists \{y_i\}_{i=1}^{p} (\text{not all zero}) \Longrightarrow \boldsymbol{a}_1 y_1 + \boldsymbol{a}_2 y_2 + \cdots + \boldsymbol{a}_p y_p = 0. \tag{2}$$

In (2), there exists at least one $y_i > 0$ (if $\forall y_i \leq 0$, multiply (2) by -1).

$$(1) - \epsilon \times (2) \Rightarrow (x_1 - \epsilon y_1)\boldsymbol{a}_1 + (x_2 - \epsilon y_2)\boldsymbol{a}_2 + \cdots + (x_p - \epsilon y_p)\boldsymbol{a}_p = \boldsymbol{b}. \tag{3}$$

By defining $\boldsymbol{y} = [y_1, \cdots, y_p, 0, \cdots, 0] \Longrightarrow \boldsymbol{A}(\boldsymbol{x} - \epsilon \boldsymbol{y}) = \boldsymbol{b}$.

$\because \ \boldsymbol{x} - \epsilon \boldsymbol{y} = [x_1 - \epsilon y_1, \cdots, x_p - \epsilon y_p, 0, \cdots, 0] \xRightarrow{\text{if } \epsilon \downarrow 0} \boldsymbol{x} - \epsilon \boldsymbol{y} \geq 0$

$\therefore \ \boldsymbol{x} - \epsilon \boldsymbol{y}$ is f.s with at most $p - 1$ positive entries.

Repeat the process $\Longrightarrow \boldsymbol{x} - \epsilon \boldsymbol{y}$ with at most $p-1$, $p-2 \cdots$ positive entries.

The number of $\boldsymbol{a}_i$'s decreases until $\boldsymbol{a}_i$'s are linear independent. $\Rightarrow$ a b.f.s.

Solution of LP

★ $M = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq 0\}$ is convex polyhedron.

extreme point v.s. basic feasible solution

$\boldsymbol{x} \in M$ is an extreme point $\iff$ $\boldsymbol{x}$ is a basic feasible solution of M.

proof. ($\Leftarrow$): let $\boldsymbol{x} \in M$ be b.f.s. Assume that

$$\boldsymbol{x} = \alpha \boldsymbol{y} + (1-\alpha)\boldsymbol{z}, \text{ for some } \alpha \in (0,1),\ \boldsymbol{y} \in M,\ \boldsymbol{z} \in M. \tag{4}$$

$\because \boldsymbol{y} \in M, \boldsymbol{z} \in M \Rightarrow \boldsymbol{y} \geq 0, \boldsymbol{z} \geq 0.$

$\because \boldsymbol{x}$ is b.f.s $\Rightarrow \boldsymbol{x}$ has at least $n-m$ zeros. (w.l.o.g) let

$$\boldsymbol{x} = [x_1, x_2, \cdots, x_m, 0, \cdots, 0] \xrightarrow{(4)} \begin{cases} \boldsymbol{y} = [y_1, y_2, \cdots, y_m, 0, \cdots, 0], \\ \boldsymbol{z} = [z_1, z_2, \cdots, z_m, 0, \cdots, 0]. \end{cases}$$

$$\because \begin{cases} \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \\ \boldsymbol{A}\boldsymbol{z} = \boldsymbol{b} \end{cases} \Longrightarrow \begin{cases} \boldsymbol{a}_1 y_1 + \boldsymbol{a}_2 y_2 + \cdots + \boldsymbol{a}_m y_m = 0, \\ \boldsymbol{a}_1 z_1 + \boldsymbol{a}_2 z_2 + \cdots + \boldsymbol{a}_m z_m = 0. \end{cases}$$

$\therefore \boldsymbol{a}_1(y_1 - z_1) + \boldsymbol{a}_2(y_2 - z_z) + \cdots + \boldsymbol{a}_m(y_m - z_m) = 0$

$\because \boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m$ are linear independent $\Longrightarrow y_i = z_i$ for all $i = 1, 2, \cdots, m$

$\therefore \boldsymbol{y} = \boldsymbol{z} \xrightarrow{(4)} \boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z} \Longrightarrow \boldsymbol{x}$ is extreme point.

Solution of LP

★ all extreme points of M $\Longleftrightarrow$ all basic feasible solutions of LP. Or equivalently, solving LP $\Longleftrightarrow$ checking all extreme points of M and find the minimal objective function value.

Example

$$\begin{cases} \max & 3x_1 + 5x_2 \\ \text{s.t.} & x_1 + 5x_2 \le 40, \\ & 2x_1 + x_2 \le 20, \\ & x_1 + x_2 \le 12 \\ & x_1 \ge 0, x_2 \ge 0. \end{cases} \xrightarrow[\text{form}]{\text{standard}} \begin{cases} \min & -3x_1 - 5x_2 \\ \text{s.t.} & x_1 + 5x_2 + x_3 = 40, \\ & 2x_1 + x_2 + x_4 = 20, \\ & x_1 + x_2 + x_5 = 12 \\ & x_i \ge 0, i = 1, 2, \cdots, 4. \end{cases}$$

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Ans: $\because \boldsymbol{A} = \begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 40 \\ 20 \\ 12 \end{bmatrix} \xrightarrow{\text{why?}}$ $\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s. $f(\boldsymbol{x}) = 0.$

motivation: check another (adjacent extreme point) b.f.s and compare the objective function value.

★ adjacent extreme point: two extreme points are adjacent if the corresponding basic columns differ by only one vector.

e.g., check an adjacent extreme point (a.e.p) by removing one of $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ from basis, and add one of $\boldsymbol{a}_1, \boldsymbol{a}_2$ into basis.

$[\boldsymbol{A}, \boldsymbol{b}] =$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s

$f(\boldsymbol{x}) = 0$

Solution of LP

e.g., check an adjacent extreme point (a.e.p) by removing one of $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ from basis, and add one of $\boldsymbol{a}_1, \boldsymbol{a}_2$ into basis.

$[\boldsymbol{A}, \boldsymbol{b}] =$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
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2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s

$f(\boldsymbol{x}) = 0$

$\xRightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [10, 0, 30, 0, 2]$ is b.f.s

$f(\boldsymbol{x}) = -30$

e.g., check an adjacent extreme point (a.e.p) by removing one of $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ from basis, and add one of $\boldsymbol{a}_1, \boldsymbol{a}_2$ into basis.

$[\boldsymbol{A}, \boldsymbol{b}] =$

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1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s

$f(\boldsymbol{x}) = 0$

$\xrightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [10, 0, 30, 0, 2]$ is b.f.s

$f(\boldsymbol{x}) = -30$

$\xrightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [8, 4, 12, 0, 0]$

$f(\boldsymbol{x}) = -44$

Solution of LP

e.g., check an adjacent extreme point (a.e.p) by removing one of $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ from basis, and add one of $\boldsymbol{a}_1, \boldsymbol{a}_2$ into basis.

$[\boldsymbol{A}, \boldsymbol{b}] =$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s

$f(\boldsymbol{x}) = 0$

$\xRightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [10, 0, 30, 0, 2]$ is b.f.s

$f(\boldsymbol{x}) = -30$

$\xRightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [8, 4, 12, 0, 0]$

$f(\boldsymbol{x}) = -44$

$\xRightarrow{a.e.p}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [5, 7, 0, 3, 0]$ is b.f.s

$f(\boldsymbol{x}) = -50$

e.g., check an adjacent extreme point (a.e.p) by removing one of $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ from basis, and add one of $\boldsymbol{a}_1, \boldsymbol{a}_2$ into basis.

$[\boldsymbol{A}, \boldsymbol{b}] =$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [0, 0, 40, 20, 12]$ is b.f.s

$f(\boldsymbol{x}) = 0$

$\overset{a.e.p}{\Longrightarrow}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [10, 0, 30, 0, 2]$ is b.f.s

$f(\boldsymbol{x}) = -30$

$\overset{a.e.p}{\Longrightarrow}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [8, 4, 12, 0, 0]$

$f(\boldsymbol{x}) = -44$

$\overset{a.e.p}{\Longrightarrow}$

$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{b}$
1	5	1	0	0	40
2	1	0	1	0	20
1	1	0	0	1	12

$\boldsymbol{x} = [5, 7, 0, 3, 0]$ is b.f.s

$f(\boldsymbol{x}) = -50$