Convex Functions

2. Functional Operations Preserving Convexity

Operations Preserving Convexity

Proposition (positive combinations)

Let $\{f_i\}_{i=1}^m$ be in Conv$\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$] and $\{t_i > 0\}_{i=1}^m$. Assume that there is a point where all f_j's are finite. Then $f := \sum_{j=1}^m t_j f_j$ is in Conv$\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$].

proof Convexity of f can be proved from the definition.
As for closedness, since $t_i > 0$ and f_i closed, we have

$$\liminf_{y\to x} t_i f_i(y) = t_i \liminf_{y\to x} f_i(y) \geq t_i f_i(x).$$

►e.g. Let $f \in \overline{\text{Conv}}\mathbb{R}^n$, $C \subset \mathbb{R}^n$ be closed convex, and $\text{dom} f \cap C \neq \emptyset$. Then $f + \iota_C$ is in $\overline{\text{Conv}}\mathbb{R}^n$. Thus,
$\inf\{f(x) : x \in C\} \iff \inf\{f(x) + \iota_C(x) : x \in \mathbb{R}^n\}$.

Proposition (supremum of convex functions)

Let $\{f_j\}_{j\in J}$ be a family of functions in Conv$\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$]. If there exists x_0 such that $\sup_j f_j(x_0) < +\infty$, then the pointwise supremum $f := \sup\{f_j : j \in J\}$ is in Conv$\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$].

proof supremum of functions corresponds to intersection of epigraphs, i.e., $\mathrm{epi} f = \cap_{j\in J} \mathrm{epi} f_j$, which preserves convexity and closedness.

▶e.g. Conjugate function. Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ (not $\equiv +\infty$) be minorized by affine function (i.e., $f \geq \langle s_0, \cdot\rangle - b$ on $\mathbb{R}^n$). Then conjugate function of f is

$$f^*(s) := \sup\{\langle s, x\rangle - f(x) : x \in \mathrm{dom} f\}.$$

$\because$ $\mathrm{dom} f \neq \emptyset$, $\therefore$ $f^*(s_0) \leq b$ and $f^*(s) > -\infty$ for all s. $\therefore$ $f^* \in \overline{\mathrm{Conv}}\mathbb{R}^n$.

▶e.g. Let $\emptyset \neq S \subset \mathbb{R}^n$ (possibly nonconvex) and

$$\varphi_S(x) = \frac{1}{2}[\|x\|^2 - d_S^2(x)],$$

where d_S is distance function to S with Euclidean norm $\|\cdot\|$. Then, φ_S is convex.

Hint: $\because$ $d_S^2(x) = \inf_{c\in S}\|x - c\|^2 = \|x\|^2 - \sup_{c\in S}[2\langle c, x\rangle - \|c\|^2]$

$\therefore$ $\varphi_S(x) = \sup\{\langle c, x\rangle - \frac{1}{2}\|c\|^2 : c \in S\}$.

$\therefore$ φ_S is pointwise supremum of affine function $\langle c, \cdot\rangle - \frac{1}{2}\|c\|^2$. $\therefore$ closed and convex.

Indeed, φ_S is the conjugate of $\frac{1}{2}\|\cdot\|^2 + \iota_S$.

Operations Preserving Convexity

Proposition (pre-composition with affine mapping)

Let $f \in \mathrm{Conv}\mathbb{R}^n$ [resp. $\overline{\mathrm{Conv}}\mathbb{R}^n$] and $A : \mathbb{R}^m \to \mathbb{R}^n$ be affine mapping such that $\mathrm{Im}A \cap \mathrm{dom}f \neq \emptyset$. Then, $(f \circ A)(x) = f(A(x))$ is in $\mathrm{Conv}\mathbb{R}^m$ [resp. $\overline{\mathrm{Conv}}\mathbb{R}^n$].

proof Clearly, $(f \circ A)(x) > -\infty$ for all x;

Besides, $\exists y = A(x) \in \mathbb{R}^n$ such that $f(y) < +\infty$.

For convexity, it suffices to plug

$$A(\alpha x + (1-\alpha)x') = \alpha A(x) + (1-\alpha)A(x')$$

into the definition of convexity.

For closedness, it comes from continuity of A when f is closed.

▶e.g. If $f \in \mathrm{Conv}\mathbb{R}^n$, take $x_0 \in \mathrm{dom} f$, $d \in \mathbb{R}^n$ and define affine mapping

$$A(t) := x_0 + td.$$

Then, $f \circ A$ indeed restricts f along the line $x_0 + \mathbb{R}d$.

Operations Preserving Convexity

Proposition (post-composition with increasing convex function)

Let $f \in \text{Conv}\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$] and $g \in \text{Conv}\mathbb{R}$ [resp. $\overline{\text{Conv}}\mathbb{R}$] be increasing. Assume that there is $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \text{dom} g$. Then $(g \circ f)(x) = g(f(x))$ is in $\text{Conv}\mathbb{R}^n$ [resp. $\overline{\text{Conv}}\mathbb{R}^n$].

proof It suffices to check the definitions of convexity and closedness.

- e.g. $g(t) := \exp t$ is convex increasing, $\therefore$ $\exp f(x)$ is [closed] convex when f is [closed] convex.
- e.g. $f : \mathbb{R}^n \to \mathbb{R}_\infty$ is called logarithmically convex if $\log f \in \text{Conv}\mathbb{R}^n$. Indeed, $\because$ $f = \exp \log f$, logarithmically convex function must be convex.
- e.g. Square of nonnegative convex function is convex. $\because$ post-compose it by $g(t) = (\max\{0, t\})^2$.

Definition (dilation of function)

For $f \in \text{Conv}\mathbb{R}^n$ and $u > 0$, the dilation function $f_u(x) := uf(x/u)$ is convex.

- $\because$ $f_u(x/u) = f(x/u)$, $\therefore$ $\text{epi} f_u = u\text{epi} f$, $\text{epi}_s f_u = u\text{epi}_s f$, $S_r(f_u) = uS_{r/u}(f)$, which implies f_u is a "dilated version" of f.

Operations Preserving Convexity

Definition (perspective of function)

The *perspective* of f, denoted by $\tilde{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_\infty$, is given by

$$\tilde{f}(u,x) := \begin{cases} uf(x/u) & \text{if } u > 0, \\ +\infty & \text{if not.} \end{cases}$$

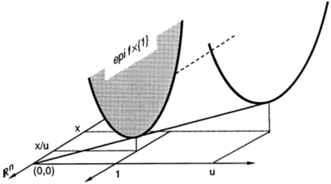
Proposition

If $f \in \mathrm{Conv}\mathbb{R}^n$, then its perspective $\tilde{f}$ is in $\mathrm{Conv}\mathbb{R}^{n+1}$.

proof Analyze $\tilde{f}$ with "geometric glasses":

$$\begin{aligned} \text{epi}\tilde{f} &= \{(u,x,r) \in \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R} : f(x/u) \le r/u\} \\ &= \{(u(1,x',r') : u > 0, (x',r') \in \text{epi } f\} \\ &= \cup_{u>0} u(\{1\} \times \text{epi } f) = \mathbb{R}_+^*(\{1\} \times \text{epi } f). \end{aligned}$$

Thus, epi$\tilde{f}$ is convex cone.



- Embed epi$f$ into $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, where the first $\mathbb{R}$ represents u; translate it horizontally by one unit; take the positive multiples of the result.
- dom$\tilde{f} = \mathbb{R}_+^*(\{1\} \times \text{dom} f)$.
- epi$\tilde{f}$ [resp. dom$\tilde{f}$] does not contain 0.

e.g. For an $x_0 \in \text{dom} f$, then $d \to f(x_0 + d) - f(x_0)$ is convex. Its perspective $r(u, d) := uf(x_0 + d/u) - f(x_0)$ is essentially the difference quotient.

Proposition (closedness of a perspective function)

Let $f \in \operatorname{Conv} \mathbb{R}^n$ *and let* $x' \in \operatorname{ridom} f$. *Then the closure* $\operatorname{cl} \tilde{f}$ *is*

$$(\operatorname{cl} \tilde{f})(u, x) = \begin{cases} uf(x/u) & \text{if } u > 0, \\ \lim_{\alpha \downarrow 0} \alpha f(x' - x + x/\alpha) & \text{if } u = 0, \\ +\infty & \text{if } u < 0. \end{cases}$$

proof For any x,

- If $u < 0$, then (u, x) is outside $\mathrm{cl}\,\mathrm{dom}\,\tilde{f}$ and $\mathrm{cl}\,\tilde{f}(u, x) = +\infty$.
- If $u \geq 0$. then

$$(\mathrm{cl}\,\tilde{f})(u, x) = \lim_{\alpha \downarrow 0} \tilde{f}\left((u, x) + \alpha[(1, x') - (u, x)]\right)$$
$$= \lim_{\alpha \downarrow 0}[u + \alpha(1 - u)] f\left(\frac{x + \alpha(x' - x)}{u + \alpha(1 - u)}\right).$$

- If $u = 1$, $\mathrm{cl}\,\tilde{f}(1, x) = \mathrm{cl}\, f(x) = f(x)$ because f is closed; if $u = 0$, we obtain our claimed relation.

- $\tilde{f}(u, \cdot)$ for $u \downarrow 0$ depends on the behaviour of f at infinity. If $x = 0$, then

$$\mathrm{cl}\,\tilde{f}(0, 0) = \lim_{\alpha \downarrow 0} \alpha f(x') = 0 \quad [f(x') < +\infty].$$

If $x \neq 0$ and dom$f$ is bounded, then $f(x' - x + x/\alpha) = +\infty$ for α small enough and cl$\tilde{f}(0, x) = +\infty$.

Analysis For two functions f_1 and f_2, let

$$C := \text{epi}\, f_1 + \text{epi}\, f_2 = \{(x_1 + x_2, r_1 + r_2) : r_j \geq f_j(x_j) \text{ for } j = 1, 2\}.$$

Under some minorization property, C has a lower-bound function ℓ_C:

$$\ell_C(x) = \inf\{r_1 + r_2 : r_j \geq f_j(x_j) \text{ for } j = 1, 2,\ x_1 + x_2 = x\}.$$

Above optimization involves (r_1, r_2, x_1, x_2). Indeed, r_j's can be eliminated.

Definition (infimal convolution)

Let $f_i : \mathbb{R}^n \to \mathbb{R}_\infty$ $(i = 1, 2)$. The infimal convolution of f_1 and f_2 is a function from $\mathbb{R}^n$ to $\mathbb{R}_\infty$, defined by

$$(f_1 \star f_2)(x) := \inf\{f_1(x_1) + f_2(x_2) : x_1 + x_2 = x\} = \inf_{y \in \mathbb{R}^n} [f_1(y) + f_2(x - y)].$$

Proposition

Let $f_i \in \text{Conv}\mathbb{R}^n$ $(i = 1, 2)$. Suppose they have common affine minorant, i.e., $\exists (s, b) \in \mathbb{R}^n \times \mathbb{R}$ such that $f_j(x) \geqslant \langle s, x \rangle - b$ for all $j = 1, 2$ and $x \in \mathbb{R}^n$. Then $(f_1 \star f_2) \in \text{Conv}\mathbb{R}^n$.

properties of infimal convolution

- $f_1 \star f_2 = f_2 \star f_1$ (commutativity)
- $(f_1 \star f_2) \star f_3 = f_1 \star (f_2 \star f_3)$ (associativity)
- $f \star \iota_{\{0\}} = f$ (existence of a neutral element in $\overline{\text{Conv}}\mathbb{R}^n$)
- $f_1 \leq f_2 \Rightarrow f_1 \star g \leq f_2 \star g$ ($\star$ preserves the order).
- ► $(f_1 \star \cdots \star f_m)(x) = \inf\{\sum_{j=1}^m f_j(x_j) : \sum_{j=1}^m x_j = x\}$. (extension of infimal convolution)

►e.g. If $\emptyset \neq C_i \subset \mathbb{R}^n$ $(i = 1, 2)$ are convex, then $\iota_{C_1} \star \iota_{C_2} = \iota_{C_1+C_2}$.

- since the sum of closed sets may not be closed, inf-convolution can preserve closed, even f_i are closed functions.

►e.g. Let $\emptyset \neq C \subset \mathbb{R}^n$ be convex and $\|\cdot\|$ be any norm. Then, $\iota_C \star \|\cdot\| = d_C$.

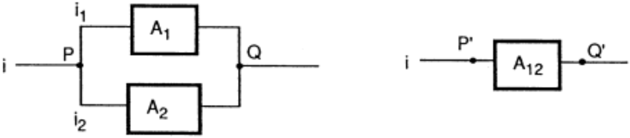
- it implies d_C is convex. It shows inf-convolution of non-closed functions (C may be not closed) can result in closed function.

►e.g. Let f be convex function minorized by affine function $g = \langle s, \cdot\rangle - b$ with slope s. Then, $f \star g = g - c$, where $c = \sup_y[\langle s, y\rangle - f(y)]$.
Indeed, $c = f^*(s)$.

▶e.g. Let f be convex function minorized by affine function. Then $f_u \star f_{u'} = f_{u+u'}$. ($\because$ $\mathrm{epi}_s f_u = u\mathrm{epi}_s f$).

- particularly, $\frac{1}{m}(f \star \cdots \star f)(x) = f(\frac{1}{m}x)$.

▶e.g. Let $f_i(x) = \frac{1}{2}\langle A_i x, x\rangle$ $(i = 1, 2)$ be quadratic forms with $A_i \succ 0$. Then, $(f_1 \star f_2)(x) = \frac{1}{2}\langle A_{12}x, x\rangle$, where $A_{12} := (A_1^{-1} + A_2^{-1})^{-1}$.



- let f_1 [resp. f_2] be the cost of producing x by some production unit U_1 [resp. U_2]. If we distribute optimally the production of a given x between U_1 and U_2, we have to compute $f_1 \star f_2$.

▶ $C_1 - C_2 = C_1 + (-C_2)$ may not an epigraph. But, $\underline{*}$ of two epigraphs is again epigraph; it is called deconvolution, i.e.,
$(f_1 \underline{\star} f_2)(x) = \sup\{f_1(x_1) - f_2(x_2) : x_1 - x_2 = x\}$.
It is supremum of convex functions. $\therefore$ it is convex if $\mathrm{epi} f_1 \underline{*} \mathrm{epi} f_2 \neq \emptyset$.
It is closed when f_1 is closed.

- Let f_1 be convex and
  - $f_2(x) = \iota_{\{x\}}(x) + r$, then $f_1 \star f_2$ is shift $\mathrm{epi}\, f_1$ vertically by r;
  - $f_2(x) = \iota_{\{x_0\}}(x)$, then $f_1 \star f_2$ is horizontal shift;
  - $f_2(x) = \iota_{B(0,r)}(x)$, then $f_1 \star f_2$ is horizontal smear;
  - $f_2(x) = sx - r$, then $f_1 \star f_2$ implies f_2 wins;
  - $f_2(x) = 1 - \sqrt{1 - \|x\|^2}$ for $x \in B(0,1)$ (ball-pen function), then $f_1 \star f_2$ translate bottom of ball-pen (i.e., 0) to each point in $\mathrm{gr}\, f_1$;
  - $f_2(x) = \frac{1}{2}\|x\|^2$, then $f_1 \star f_2$ is Moreau-Yosida regularization of f.

- Classical convolution:

$$(F_1 * F_2)(x) := \int_{\mathbb{R}^n} F_1(y)F_2(x-y)dy, \ \forall x \in \mathbb{R}^n.$$

Generalized convolution: for $F_i \geq 0$ and $p > 0$,

$$(F_1 *_p F_2)(x) := \left\{\int_{\mathbb{R}^n} [F_1(y)F_2(x-y)]^p dy\right\}^{1/p}.$$

let $p \to +\infty$, the right-hand side $= \sup_y F_1(y)F_2(x-y)$.

By take $F_i := \exp(-f_i)$ $(i = 1, 2)$, we have

$$(F_1 *_\infty F_2)(x) = \sup_y e^{-f_1(y)-f_2(x-y)} = e^{-\inf_y [f_1(y)+f_2(x-y)]}.$$

Image of function under linear mapping

Definition (value/marginal/perturbation/primal function)

For constrained optimization in variable u, then $f(x) = \inf_{u \in U}\{\varphi(u) : c(u) \leqslant x\}$, where x is perturbation.

Definition (image function)

Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be linear operator and $g : \mathbb{R}^m \to \mathbb{R}_\infty$. The image of g under A, denoted by $Ag : \mathbb{R}^n \to \mathbb{R}_\infty$, is defined as

$$(Ag)(x) := \inf\{g(y) : Ay = x\}.$$

►e.g. If $g = \iota_C$ with $\emptyset \neq C \in \mathbb{R}^m$, then $(Ag)(x) = \begin{cases} 0 & \text{if } x \in A(C) \\ +\infty & \text{otherwise} \end{cases} = \iota_{A(C)}$.

it is convex when C is convex.

Theorem

Let $g \in \mathrm{Conv}\mathbb{R}^m$. Assume that, for all $x \in \mathbb{R}^n$, g is bounded below on inverse image $A^{-1}(x) = \{y \in \mathbb{R}^m : Ay = x\}$. Then $Ag \in \mathrm{Conv}\mathbb{R}^n$.

proof By assumption, $Ag(x) > -\infty$ for all x; and $(Ag)(x) < +\infty$ with $x = Ay$ and $y \in \text{dom}g$.

Define $A'(y, r) := (Ay, r) \in \mathbb{R}^m \times \mathbb{R}$.

The set $A'(\text{epi}g) =: C$ is convex. The lower-bound function is: for given $x \in \mathbb{R}^n$,

$$\inf_r \{r : (x, r) \in C\} = \inf_{y,r} \{r : Ay = x,\ g(y) \le r\} = \inf_y \{g(y) : Ay = x\} = (Ag)(x)$$

which proves the convexity of $Ag = \ell_C$.

- Usually, $A^{-1}(x)$ is affine manifold. $Ag(x)$ selects one giving the smallest value of g. Particularly, if A is invertible, $Ag = g \circ A^{-1}$.
- $\text{epi}(Ag)$ is the epigraphical hull of the inverse image $A^{-1}(\text{epi}\, g)$ (a convex set).

Corollary (perturbation function)

Let $U = \mathbb{R}^p$, $\varphi \in \text{Conv}\,\mathbb{R}^p$, $X = \mathbb{R}^n$ is equipped with the canonical basis, the mapping c has its components $c_j \in \text{Conv}\,\mathbb{R}^p$ $(j = 1, \dots, n)$. Suppose that the optimal value is $> -\infty$ for all $x \in \mathbb{R}^n$, and $\text{dom}\,\varphi \cap \text{dom}\,c_1 \cap \cdots \cap \text{dom}\,c_n \neq \emptyset$. Then the value function

$$v_{\varphi,c}(x) := \inf\{\varphi(u) : c_j(u) \leqslant x_j,\ j = 1, \dots, n\} \in \text{Conv}\,\mathbb{R}^n.$$

proof Assume $v_{\varphi,c}(x) > -\infty$ for all x.

Take $u_0 \in \operatorname{dom}\varphi \cap \operatorname{dom} c_1 \cap \cdots \cap \operatorname{dom} c_n$ and set $M := \max_i c_i(u_0)$;

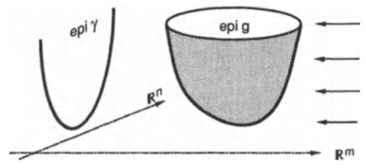
then take $x_0 := (M, \ldots, M) \in \mathbb{R}^n$, so that $v_{\varphi,c}(x_0) \leq \varphi(u_0) < +\infty$. Knowing that $v_{\varphi,c}$ is an image-function, we just have to prove the convexity of $C = \{y = (u, v) \in \mathbb{R}^m : c(u) \leqslant v\}$; indeed, it comes from convexity of c_i.

▶e.g. Let $f_i \in \operatorname{Conv}\mathbb{R}^n$ $(i = 1, 2)$, $g(x_1, x_2) = f_1(x_1) + f_2(x_2)$, and $A(x_1, x_2) = x_1 + x_2$. Then, $Ag = f_1 \star f_2$.

it shows that an image of a closed function need not be closed.

Definition (marginal function)

Let $g \in \operatorname{Conv}(\mathbb{R}^n \times \mathbb{R}^m)$. Then $\gamma(x) := \inf\{g(x, y) : y \in \mathbb{R}^m\}$.



Proposition (convex hull of function: co g)

Let $g : \mathbb{R}^n \to \mathbb{R}_\infty$ (not $\equiv +\infty$), be minorized by affine function $g(x) \geqslant \langle s, x \rangle - b$. Then, the following $\{f_i\}_{i=1}^3$ are convex and coincide on $\mathbb{R}^n$:

$$f_1(x) := \inf\{r : (x, r) \in \operatorname{co} \operatorname{epi} g\},$$
$$f_2(x) := \sup\{h(x) : h \in \operatorname{Conv} \mathbb{R}^n,\ h \leqslant g\},$$
$$f_3(x) := \ \inf\left\{\sum_{i=1}^k \alpha_i g(x_i) : k = 1, 2, \ldots, \alpha \in \Delta_k,\ x_i \in \operatorname{dom} g,\ \sum_{i=1}^k \alpha_i x_i = x\right\}.$$

Proposition (closed convex hull of function: $\overline{\text{co}}\ g$)

Let g as in the above Proposition. Then, the following $\{\bar{f}_i\}_{i=1}^3$ are closed convex and coincide on $\mathbb{R}^n$:

$$\bar{f}_1(x) := \inf\{r : (x, r) \in \overline{\operatorname{co}} \operatorname{epi} g\},$$
$$\bar{f}_2(x) := \sup\{h(x) : h \in \overline{\operatorname{Conv}}\mathbb{R}^n,\ h \leqslant g\},$$
$$\bar{f}_3(x) := \sup\{\langle s, x \rangle - b : \langle s, y \rangle - b \leqslant g(y),\ \forall y \in \mathbb{R}^n\}$$

Proposition (convex hull of function: co g)

Let $g_i \in$ Conv$\mathbb{R}^n$ $(i = 1, \ldots, m)$, all minorized by the same affine function. Then, the convex hull of their infimum is defined by

$$[\mathrm{co}(\min_i g_i)](x) := \inf \left\{ \sum_{i=1}^{k} \alpha_i g_i(x_i) : \alpha \in \Delta_k,\ x_i \in \operatorname{dom} g_i,\ \sum_{i=1}^{k} \alpha_i x_i = x \right\}.$$

▶e.g. Let $\{(x_i, b_i)\}_{i=1}^m$ be points in $\mathbb{R}^n \times \mathbb{R}$, $g_i(x) = \begin{cases} b_i & \text{if } x = x_i \\ +\infty & \text{if not.} \end{cases}$

Then $f := \mathrm{co}(\min g_i) = \overline{\mathrm{co}}(\min g_i)$ is polyhedral function, specifically,

$$f(x) = \begin{cases} \min \left\{ \sum_{i=1}^{m} \alpha_i b_i : \alpha \in \Delta_m,\ \sum_{i=1}^{m} \alpha_i x_i = x \right\}, & \text{if } x \in \mathrm{co}\{x_1, \ldots, x_m\}, \\ +\infty, & \text{if not} \end{cases}$$

$$= \begin{cases} \min \left\{ b^\top \alpha : \alpha \in \Delta_m,\ A\alpha x \right\}, & \text{if } x \in \mathrm{co}\{x_1, \ldots, x_m\}, \\ +\infty, & \text{if not.} \end{cases}$$