Convex Functions

1. Basic Definitions and Examples

Convex Function

Definition

Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be convex. A function $f : C \to \mathbb{R}$ is convex on C if

$$f\left(\alpha x + (1-\alpha)x'\right) \leqslant \alpha f(x) + (1-\alpha) f\left(x'\right), \ \forall (x, x') \in C \times C, \ \forall \alpha \in]0,1[. \quad (1)$$

- f is strictly convex on C if "$<$" holds for all $x \neq x'$.
- f is strongly convex if there exists $c > 0$ such that

$$f\left(\alpha x + (1-\alpha)x'\right) \leqslant \alpha f(x) + (1-\alpha) f\left(x'\right) - \frac{1}{2} c\alpha(1-\alpha) \left\|x - x'\right\|^2 \quad (2)$$

for all $(x, x') \in C \times C$ and $\alpha \in]0,1[$.

▶ To guarantee the left-hand side of (1) makes sense, C must be convex.

Proposition

f is c-strongly convex on C if and only if $f - \frac{c}{2}\|\cdot\|^2$ is convex on C.

proof By applying the definition of convexity to $f - \frac{c}{2}\|\cdot\|^2$, we have

$$f\left(\alpha x+(1-\alpha) x^{\prime}\right)-\frac{1}{2} c\left\|\alpha x+(1-\alpha) x^{\prime}\right\|^2$$
$$\leqslant \alpha f(x)+(1-\alpha) f\left(x^{\prime}\right)-\frac{1}{2} c[\alpha\|x\|^2+(1-\alpha)\left\|x^{\prime}\right\|^2].$$

Definition

$f:\mathbb{R}^n \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ (not identically $+\infty$) is said to be convex if

$$f\left(\alpha x+(1-\alpha) x^{\prime}\right) \leqslant \alpha f(x)+(1-\alpha) f\left(x^{\prime}\right), \forall\left(x, x^{\prime}\right) \in \mathbb{R}^n \times \mathbb{R}^n, \forall \alpha \in]0,1[$$

The class of such functions is denoted by $\mathrm{Conv}\mathbb{R}^n$.

To realize the equivalence between two definitions, extend f by defining $f(x) := +\infty$ for $x \notin C$, which is in Conv $\mathbb{R}^n$.

Definition

Domain (or effective domain) of f is the nonempty set
dom$f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$.
Graph of a function is the set of couples gph$f = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \mathbb{R}^n\}$.

Definition (epigraph and sublevel-set)

Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ (not identically $+\infty$). The epigraph of f is the nonempty set epi$f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : r \geqslant f(x)\}$.

- Strict epigraph $\mathrm{epi_s}\, f$: replace $\geqslant$ with $>$ (word "strict" is irrelevant to strict convexity).
- Sublevel-sets: $\mathrm{S}_r(f) := \{x \in \mathbb{R}^n : f(x) \leqslant r\}$. Or equivalently,

$$(x, r) \in \mathrm{epi}\, f \iff x \in \mathrm{S}_r(f) \tag{3}$$

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$ (not identically $+\infty$). The three properties below are equivalent:

(i) *f is convex in the sense of convex;*

(ii) *its epigraph is a convex set in $\mathbb{R}^n \times \mathbb{R}$;*

(iii) *its strict epigraph is a convex set in $\mathbb{R}^n \times \mathbb{R}$.*

- f is concave $\iff$ $-f$ is convex, i.e., the hypograph of f is convex.

- $f \in \mathrm{Conv}\mathbb{R}^n \implies S_r(f)$ is convex for all r.

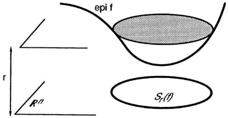
  But, all $S_r(f)$ are convex $\not\Rightarrow f$ is convex (indeed, it is called quasi-convex).
- How to construct $S_r(f)$?

  cut epi$f$ by horizontal blade

  $\rightarrow$form the intersection epi $f \cap (\mathbb{R}^n \times \{r\})$

  $\rightarrow$project down to $\mathbb{R}^n \times \{0\}$

  $\rightarrow$change the space from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$.



- $\mathrm{dom}\, f = \cup_{r \in \mathbb{R}} \mathrm{S}_r(f)$ (i.e., a nested family); $\mathrm{dom}\, f =$ project epi f onto $\mathbb{R}^n$.

Theorem (Jensen Inequality)

Let $f \in \mathrm{Conv}\,\mathbb{R}^n$, $\{x_i\}_{i=1}^k$ be points in $\mathrm{dom}\, f$ and $\{\alpha_i\}_{i=1}^k$ be scalars in unit simplex of $\mathbb{R}^k$. Then, $f\left(\sum_{i=1}^k \alpha_i x_i\right) \leqslant \sum_{i=1}^k \alpha_i f(x_i)$.

proof The points $\{(x_i, f(x_i))\}_{i=1}^k \subset \mathbb{R}^n \times \mathbb{R}$ are clearly in epi$f$ (a convex set).

Thus, the convex combination

$\sum_{i=1}^m \alpha_i (x_i, f(x_i)) = (\sum_i \alpha_i x_i, \sum_i \alpha_i f(x_i)) \in \mathrm{epi} f$.

- $f \in \operatorname{Conv}\mathbb{R}^n \iff \operatorname{epi} f$ is convex. Thus, $f(x) = \inf\limits_{(x,r)\in\operatorname{epi} f} r$;
- Closedness of epigraph: $\because$ affepi$f$ $\supset$vertical lines $\{x\} \times \mathbb{R}$, where $x \in \operatorname{dom} f$.
  $\therefore$ $\operatorname{epi} f$ cannot be open. $\therefore$ riepi$f$ cannot be an epigraph.

Proposition (Let $f \in \operatorname{Conv}\mathbb{R}^n$)

The $\operatorname{riepi} f$ *is the union over* $x \in \operatorname{ri}\operatorname{dom} f$ *of the open half-lines with bottom endpoints at* $f(x)$, *i.e.,*

$$\textit{riepi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{ridom} f,\ r > f(x)\}.$$

- *riepi$f$ $\neq$ epi$_s f$ ($\because$ side-effect on ribd*$\operatorname{dom} f$.

- linear function: $f(x) = \langle s, x\rangle$. Then, epi$f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid r \geqslant \langle s, x\rangle\}$.
- Affine function: $f(x) = f(x_0) + \langle s, x - x_0\rangle$. Then

$$\begin{aligned}\operatorname{epi} f &= \{(x, r) : r \geqslant f(x_0) + \langle s, x - x_0\rangle\}\\ &= \{(x, r) : \langle s, x\rangle - r \leqslant \langle s, x_0\rangle - f(x_0)\}\end{aligned}$$

  i.e., epi$f$ is closed halfspace with $(s, -1) \in \mathbb{R}^n \times \mathbb{R}$ as normal vector.

Proposition

If $f \in \operatorname{Conv} \mathbb{R}^n$, then f is minorized by affine functions. More precisely, $\forall x_0 \in \operatorname{ridom} f$, $\exists s$ in the subspace parallel to affdom$f$ such that

$$f(x) \geqslant f(x_0) + \langle s, x - x_0 \rangle \text{ for all } x \in \mathbb{R}^n$$

i.e., the affine function can be forced to coincide with f at x_0.

- convex epigraph is supported by nonvertical hyperplane;
- convex function, having affine minorant, is bounded from below on bounded set.

Definition (Closed Convex Functions)

A function f is lower semicontinuous (l.s.c.) if $\liminf_{y \to x} f(y) \geqslant f(x)$, $\forall x \in \mathbb{R}^n$.

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}_\infty$. The following three properties are equivalent:

(i) *f is l.s.c. on $\mathbb{R}^n$;*

(ii) *epi f is a closed set in $\mathbb{R}^n \times \mathbb{R}$;*

(iii) *$\mathrm{S}_r(f)$ are closed (possibly empty) for all $r \in \mathbb{R}$.*

proof (i)⇒(ii): Let $\{(y_k, r_k)\}_k^\infty$ be a sequence of $\mathrm{epi} f$ converging to (x, r). Since $f(y_k) \leqslant r_k$ for all k, the l.s.c. gives

$$r = \lim r_k \geqslant \liminf f(y_k) \geqslant \liminf_{y \to x} f(y) \geqslant f(x), \text{ i.e. } (x, r) \in \mathrm{epi} f.$$

(ii)⇒(iii): Since $\mathrm{S}_r(f) = \mathrm{epi} f \cap [\mathbb{R}^n \times \{r\}]$. Thus, claim is true because of closedness of $\mathrm{epi} f$.

(iii)⇒(i): if f is not l.s.c. at some x, then there is a subsequence of $\{y_k\}$ converging to x such that $f(y_k)$ converges to $\rho < f(x) \leqslant +\infty$. Pick $r \in]\rho, f(x)[$: for k large enough, $f(y_k) \leqslant r < f(x)$;
$\therefore$ $\mathrm{S}_r(f)$ contains the tail of $\{y_k\}$ but not its limit x. Thus, $\mathrm{S}_r(f)$ is not closed.

Definition (Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$)

f is closed $\iff$ f is l.s.c $\iff$ $\mathrm{epi} f$ is closed $\iff$ $S_r(f)$ is closed for all r.

Definition (Closure or l.s.c. hull of a function)

The closure of f, denoted by $\mathrm{cl}\, f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, is defined as

$$\mathrm{cl}\, f(x) := \liminf_{y \to x} f(y),\ \forall x \in \mathbb{R}^n, \text{ or equivalently, } \mathrm{epi}(\mathrm{cl}\, f) := \mathrm{cl}(\mathrm{epi}\, f). \tag{4}$$

The closure is complicated to obtain because the gap between f and $\operatorname{cl} f$ may be impossible to control. However, the convexity makes the things easier because

1. convex function is minorized by affine function; closing it cannot yield $-\infty$.
2. it reduces to 1D setting owing to the following radial construction of $\operatorname{cl} f$.

Proposition

Let $f \in \operatorname{Conv} \mathbb{R}^n$ *and* $x' \in \operatorname{ridom} f$. *Then*

$$\operatorname{cl} f(x) = \lim_{t \downarrow 0} f(x + t(x' - x)), \ \forall x \in \mathbb{R}^n. \tag{5}$$

proof $\because$ $x_t := x + t(x' - x)$ and $\lim_t \downarrow 0 x_t = x$, we have

$$(\mathrm{cl}\, f)(x) \leqslant \liminf_{t \downarrow 0} f(x + t(x' - x))$$

We now prove the converse inequality by showing that

$$\limsup_{t \downarrow 0} f(x + t(x' - x)) \leqslant r,\ \forall r \geqslant (\mathrm{cl}\, f)(x)$$

let $(x, r) \in \mathrm{epi}(\mathrm{cl}\, f) = \mathrm{cl}$ (epi f) and pick $r' > f\,(x')$. $\therefore$ $(x', r') \in$ ri epi f. we have

$$(x', r') + (1 - t)(x, r) \in \mathrm{riepi}\, f \subset \text{epi} f,\ \forall t \in]0, 1].$$

It means $f(x + t(x' - x)) \leqslant tr' + (1 - t)r$ for all $t \in]0, 1]$, which completes the proof by letting $t \downarrow 0$.

Proposition

If $f \in \operatorname{Conv} \mathbb{R}^n$, then: ❶ $\operatorname{cl} f \in \operatorname{Conv} \mathbb{R}^n$*;* ❷ $\operatorname{cl} f$ *and* f *coincide on ri* $\operatorname{dom} f$*.*

proof Since epi cl$f$ = cl epi f is convex and $\operatorname{cl} f \leqslant f \not\equiv +\infty$; Proposition 1.2.1 guarantees $\operatorname{cl} f(x) > -\infty$ for all x. Thus, ❶ holds.

Suppose $x \in \operatorname{ri} \operatorname{dom} f$. Then $\varphi(t) = f(x + td)$ is continuous at $t = 0$.

By Proposition 1.2.5 that $\operatorname{cl} f$ coincides with f on ri dom f.

- A finite-valued convex function with dom $f = \mathbb{R}^n$ is l.s.c. (indeed, locally Lipschitzian).
- $\overline{\operatorname{Conv}} \mathbb{R}^n$: all closed convex functions on $\mathbb{R}^n$.

Proposition (outer construction of l.s.c function)

The closure of $f \in \operatorname{Conv} \mathbb{R}^n$ *is the supremum of all affine minoriant of* f*, i.e.,*

$$\operatorname{cl} f(x) = \sup_{(s,b) \in \mathbb{R}^n \times \mathbb{R}} \{\langle s, x\rangle - b : \langle s, y\rangle - b \leqslant f(y) \text{ for all } y \in \mathbb{R}^n\}. \tag{6}$$

proof A closed half-space containing epi f is defined by by

$$\langle s, x\rangle + \alpha r \leqslant b, \ \forall (x, r) \in \operatorname{epi} f, \text{ where } 0 \neq (s, \alpha) \in \mathbb{R}^n \times \mathbb{R}, b \in \mathbb{R}. \tag{7}$$

Let $\Sigma \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ be index-set of triples $\sigma = (s, \alpha, b)$, with half-space

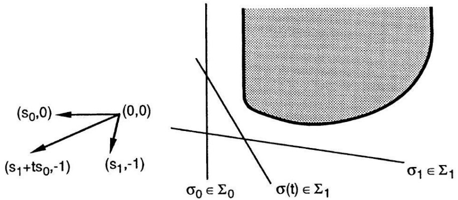
$$H_\sigma^- := \{(x, r) : \langle s, x\rangle + \alpha r \leqslant b\} \tag{8}$$

In other words, $\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f) = \cap_{\sigma \in \Sigma} H_\sigma^-$. (7) implies $\alpha \leqslant 0$ (let $r \to +\infty$). $\alpha = 0$ and $\alpha = -1$ suffice: Σ can be partitioned in

$$\Sigma_1 := \{(s, -1, b) : (1.2.10) \text{ holds with } \alpha = -1\},$$
$$\Sigma_0 := \{(s, 0, b) : (1.2.10) \text{ holds with } \alpha = 0\}.$$

Indeed, Σ_1 corresponds to affine functions minorizing f and Σ_0 to closed half-spaces of $\mathbb{R}^n$ containing dom f.

$$H_{\sigma_0}^- \cap H_{\sigma_1}^- = \cap_{t \geqslant 0} H_{\sigma(t)}^- =: H^-.$$



Given $\emptyset \neq S \subset \mathbb{R}^n$

The indicator function $\iota_S : \mathbb{R}^n \to \mathbb{R}_\infty$ and support function $\sigma_S : \mathbb{R}^n \to \mathbb{R}_\infty$ are

$$\iota_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if not,} \end{cases} \qquad \sigma_S(x) := \sup\{\langle s, x\rangle : s \in S\}.$$

- ι_S is [closed] convex $\iff$ S is [closed] convex. ($\because$ epi$\iota_S = S \times \mathbb{R}^+$).
- If $f \in \text{Conv}\mathbb{R}^n$, $\emptyset \neq C$ is convex, and dom$f \cap C \neq \emptyset$, then $$\varphi(x) = f + \iota_C = \begin{cases} f(x) & \text{if } x \in C \\ +\infty & \text{if not} \end{cases}$$ is convex. Furthermore, φ is closed if f and C are closed.
- epiσ_S is closed convex cone for all S, even if S is nonconvex. ($\because$ p.h.)
- dom$\sigma_S = \{a \in \mathbb{R}^n : \exists r \text{ such that } \langle s, a\rangle \leqslant r,\ \forall s \in S\}$ is convex cone.

Let $(s_1, b_1), \ldots, (s_m, b_m)$ be elements in $\mathbb{R}^n \times \mathbb{R}$

Piecewise affine function is $\check{f}(x) := \max\{\langle s_j, x\rangle - b_j : j = 1, \ldots, m\}$.

- $\mathbb{R}^n$ is divided into ($\leqslant m$) regions in which $\check{f}$ is affine: the j_0^{th} region (possibly empty) is closed convex polyhedron

$$\{x \in \mathbb{R}^n : \langle s_{j_0}, x\rangle - b_{j_0} \geqslant \langle s_j, x\rangle - b_j \text{ for } j = 1, \ldots, m\}.$$

Definition (polyhedral function)

f is polyhedral function if

$$\operatorname{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \langle s_j, x\rangle + \alpha_j r \leqslant b_j,\ j \in J\},$$

where J is finite set, and $(s_j, \alpha_j, b_j) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $(s_j, \alpha_j) \neq 0$.

- f is polyhedral function $\Longleftrightarrow$ $\check{f} + \iota_P$, where $\check{f}$ is piecewise affine and P is closed convex polyhedron.

norm and distance: let $\emptyset \neq C \subseteq \mathbb{R}^n$ be convex and $\|\cdot\|$ be any norm

The distance function $\mathrm{d}_C(x) := \inf\{\|y - x\| : y \in C\}$.

- d_C is continuous convex function.
  Indeed, take sequences $\{y_k\}$, $\{y'_k\}$ such that

$$\lim_{k\to\infty} \|y_k - x\| = \mathrm{d}_C(x) \text{ and } \lim_{k\to\infty} \|y'_k - x'\| = \mathrm{d}_C(x').$$

  let $\alpha \in]0,1[$, $z_k := \alpha y_k + (1-\alpha)y'_k \in C$, and pass to the limit for $k \to \infty$ in

$$\mathrm{d}_C(\alpha x + (1-\alpha)x') \leqslant \|z_k - \alpha x - (1-\alpha)x'\| \leqslant \alpha\|y_k - x\| + (1-\alpha)\|y'_k - x'\|.$$

- $\mathrm{d}_C \equiv \mathrm{d}_{\mathrm{cl}C} = \mathrm{d}_{\mathrm{ri}C}$.
- $\mathrm{d}_C = 0$ on $\mathrm{cl}\, C$. The following variant distinguishes between int$C$ and $\mathrm{bd}C$

$$\mathrm{D}_C(x) := \begin{cases} \mathrm{d}_C(x) & \text{if } x \in C^c \\ -\mathrm{d}_{C^c}(x) & \text{if } x \in C \end{cases}$$

  where C^c is the complement of C. If C and C^c are nonempty, then D_C is convex, finite everywhere, and that

$$\begin{aligned} \mathrm{int}\, C &= \{x \in \mathbb{R}^n : \mathrm{D}_C(x) < 0\}, \\ \mathrm{bd}C &= \{x \in \mathbb{R}^n : \mathrm{D}_C(x) = 0\}, \\ (\mathrm{cl}\, C)^c &= \{x \in \mathbb{R}^n : \mathrm{D}_C(x) > 0\}. \end{aligned}$$

quadratic form

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear operator. Then, the quadratic function $f(x) := \frac{1}{2}\langle Ax, x\rangle$ is convex $\iff$ $A \succeq 0$.

- Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ be eigenvalues of A. Rayleigh inequality yields $\lambda_n\|x\|^2 \leqslant \langle Ax, x\rangle \leqslant \lambda_1\|x\|^2$ for all $x \in \mathbb{R}^n$. Thus,

$$f(\alpha x + (1-\alpha)x') \leqslant \alpha f(x) + (1-\alpha) f(x') - \frac{1}{2}\lambda_n \alpha(1-\alpha)\|x - x'\|^2.$$

  $\therefore$ f is λ_n-strongly convex $\iff$ $A \succ 0$.
- Sublevel-sets $\mathrm{S}_r(f) = \{x \in \mathbb{R}^n : \frac{1}{2}\langle Ax, x\rangle \leqslant r\}$ are concentric ellipsoid, and $\mathrm{S}_{\kappa r}(f) = \sqrt{\kappa}\mathrm{S}_r(f)$.
- $\mathrm{S}_r(f)$ is degenerated () if $A \succeq 0$. It contains the subspace $\ker A$. It is an elliptic cylinder.
- $\mathrm{S}_r(f)$ is a neighborhood of 0, i.e., $\mathrm{S}_r(f) \supset B(0, \varepsilon)$ whenever $\lambda_1 \varepsilon^2 \leqslant 2r$.

sum of largest eigenvalues: let S_n be $n \times n$ symmetric matrices

For $A \in \mathrm{S}_n$, the eigenvalues of A are $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. For any $m \leqslant n$, let

$$f_m(A) := \sum_{j=1}^{m} \lambda_j(A).$$

- f_m is finite valued function. With inner product $\langle A, B\rangle := \mathrm{tr}(AB)$ in S_n, then

  $$f_m(A) = \sup\{\langle QQ^\top, A\rangle : Q \in \Omega\}, \text{ where } \Omega := \{Q : Q^\top Q = I_m\}.$$

  Indeed, $\because \Omega$ is compact, the supremum is attained at Q formed with the normalized eigenvectors of $\lambda_1, \dots, \lambda_m$.
- f_m is convex ($\because$ supremum of linear functions on S_n).
  e.g., $f_1(A) = \lambda_1$ (convex function); $f_n(A) = \mathrm{tr}(A)$ (linear function);
- however, $f_n - f_m = \sum_{i=m+1}^{m} \lambda_i$ is concave function.

volume of ellipsoid

Let $f(A) := \begin{cases} \log(\det A^{-1}) & \text{if } A \succ 0, \\ +\infty & \text{if not} \end{cases}$

$\mathrm{dom} f = \{A \in \mathrm{S}_n : \lambda_n > 0\}$ is open convex cone.

- f is convex because of the fact

$$\det[\alpha A + (1-\alpha)B] \geqslant (\det A)^{\alpha}(\det B)^{1-\alpha}, \quad \forall A, B \in S_n^{++}, \ \forall \alpha \in]0,1[.$$

- For $A \in S_{++}^n$, volume of ellipsoid $E_A = \{x \in \mathbb{R}^n : x^\top Ax \leqslant 1\}$ is (neglect positive scaling)

$$\text{vol}(E_A) = \sqrt{\det A^{-1}}$$

- $\because \operatorname{dom} f$ is open, $\therefore$ ridom$f$ =intdom$f$ =dom$f$.
- f is l.s.c. on dom$f$. Hint: by contradiction, if $\lim_{k\to\infty} A_k = A \not\succ 0$, by the continuity of concave function $\lambda_n(\cdot)$ that $A \succeq 0$ and $\lim_{k\to\infty} \lambda_n(A_k) = 0$. $\therefore f(A_k) \to +\infty$. Thus, f is closed.

epigraphical hull and lower-bound function of a convex set

Let $\emptyset \neq C \subset \mathbb{R}^n \times \mathbb{R}$ be convex. When is C the epigraph of some $f \in \operatorname{Conv}\mathbb{R}^n$?

Analysis:

1. $f(x) > -\infty$ for all x implies C should contain no vertical downward half line
$$\{r \in \mathbb{R} : (x, r) \in C\} \text{ is minorized for all } x \in \mathbb{R}^n \tag{9}$$
2. C should be unbounded from above, i.e.,
$$(x, r) \in C \implies (x, r') \in C,\ \forall r' > r. \tag{10}$$
3. C should have a "closed bottom", i.e.
$$[(x, r') \in C \text{ and } r' \downarrow r] \implies (x, r) \in C \tag{11}$$

Proposition

A set $\emptyset \neq C$ satisfying (9)-(11) is an epigraph of f. Furthermore, f is convex if C is convex.

- If C satisfying (9)-(10) but with "open bottom", i.e.
$$(x, r) \in C \implies (x, r - \varepsilon) \in C \text{ for some } \varepsilon = \varepsilon(x, r) > 0,$$
then C is a strict epigraph.
- [strict] epigraph is union of closed [open] upward half-lines.

Given set C, how to make an epigraph?

Claim: epigraphical hull of C is the smallest epigraph containing C.

1. force (10) by stuffing above C: for $(x,r) \in C$, add to C all (x,r') with $r' > r$.
2. force (11) by closing bottom of C: put (x,r) in C whenever $(x,r') \in C$ with $r' \to r$.

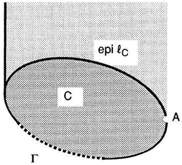
- Operations 1-2 amount to constructing the

$$\text{lower-bound function of } C: \quad \ell_C(x) := \inf\{r \in \mathbb{R} : (x,r) \in C\}. \tag{12}$$

epiℓ_C is the epigraphical hull of C;
$\ell_C(x) > -\infty$ for all x iff C satisfies (9).

- (see figure) point A and curve Γ are not in C. But, it holds

$$\mathrm{epi_s}\ell_C \subset C + \{0\} \times \mathbb{R}^+ \subset \text{ epi } \ell_C \subset \mathrm{cl}\left(C + \{0\} \times \mathbb{R}^+\right) \tag{13}$$



Theorem

Let $\emptyset \neq C \in \mathbb{R}^n \times \mathbb{R}$ satisfying (9). ❶ if C is convex, then $\ell_C \in \operatorname{Conv} \mathbb{R}^n$; ❷ if C is closed convex, then $\ell_C \in \overline{\operatorname{Conv}}\mathbb{R}^n$.

proof ❶ $\forall \varepsilon > 0$, $\alpha \in]0,1[$ and $(x_i, r_i) \in C$ such that $r_i \leqslant \ell_C(x_i) + \varepsilon$ for $i = 1, 2$.

$\because C$ is convex, $\therefore (\alpha x_1 + (1-\alpha)x_2, \alpha r_1 + (1-\alpha)r_2) \in C$,

$$\therefore \ \ell_C(\alpha x_1 + (1-\alpha)x_2) \leqslant \alpha r_1 + (1-\alpha) r_2 \leqslant \alpha \ell_C(x_1) + (1-\alpha)\ell_C(x_2) + \varepsilon.$$

This implies ℓ_C is convex because $\varepsilon > 0$ is arbitrary.

❷ Let $\{(x_k, \rho_k)\}_k \subset$ epi ℓ_C be a sequence converging to (x, ρ);

By the definition of ℓ_C, we select a r_k such that $(x_k, r_k) \in C$ and

$$\ell_C\left(x_k\right) \leqslant r_k \leqslant \ell_C\left(x_k\right) + \frac{1}{k} \leqslant \rho_k + \frac{1}{k}. \tag{14}$$

It implies that $\{r_k\}$ is bounded from above.

Also, if ℓ_C is convex, then there exists affine function minorizing ℓ_C. $\therefore \{r_k\}$ is bounded from below.

Extracting a subsequence $r_k \to r$. If C is closed, $(x, r) \in C$, $\therefore \ell_C(x) \leqslant r$; but pass to the limit in (14) to see that $r \leqslant \rho$.